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High convergence order methods with no derivatives for equations

Samundra Regmi[1], Ioannis K. Argyros [2], Gagan Deep[3]

[1] *Department of Mathematics*
University of Houston, Houston, TX 77204, USA
[2] *Department of Computing and Mathematical Sciences*
Cameron University, Lawton, OK 73505, USA
[3] *Department of Mathematics*
Hans Raj Mahila Mahavidyalaya, Jalandhar-144008, Punjab, India

Анотація. Багато задач з різних галузей науки формулюються як рівняння або системи рівнянь в абстрактних просторах, таких як евклідовий багатовимірний простір, простір Гільберта або Банаха, тощо. Більшість з цих рівнянь містять оператори, що є недиференційованими. Саме тому була розроблена спеціальна методологія для розв'язання таких рівнянь. Однак вона також може бути застосована для розв'язування рівнянь з диференційованими операторами. Для ілюстрації використовується конкретний метод, через який описується принцип роботи методології. Водночас ця методологія може бути застосована до інших методів, що залучають обернені лінійні оператори. Експерименти підтверджують дану теорію.

Abstract. Many applications from diverse disciplines are formulated as an equation or system of equations in abstract spaces such as Euclidean multidimensional, Hilbert or Banach to mention a few, a plethora of this equations are nondifferentiable. That is why a methodology is developed to handle in particular the solutions of such equations. However the methodology also applies to solve the differentiable equations. A particular method is utilized as a sample via which the method is described. However, the same methodology can be used on other methods utilizing the inverses of linear operators. Experiments further demonstrate the theory.

1 Introduction

Let H be a Fréchet differentiable operator mapping from a Banach space B into B, and D be an open convex subset of B. Obtaining a solution $\delta \in D$ of the nonlinear equation of the form

$$H(x) = 0 \tag{1.1}$$

has numerous applications in multiple disciplines as such problems are formulated as an equation like (1.1) using Mathematical Modeling [11, 21]. The iterative methods provide a way to handle non-analytic and complex functions thus approximating the solution δ of (1.1) which is rarely attainable in analytic form. No convergence, slow convergence, divergence and inefficiency are other issues that researchers using iterative methods must contend with. There is an extensive literature on the convergence of iterative methods motivated by algebraic or geometrical considerations [11, 21]. As a result, researchers worldwide are continuously striving to create higher order iterative methods

Key words: Steffensen-type Method, Convergence, Banach space, ω-continuity, Non differentiable equation

[2–5, 7, 8, 10, 12, 13, 17, 20].

In particular, we investigate the convergence of the seventh order method with no derivatives developed by Sharma and Arora [18] which is defined for each $m = 0, 1, 2, \dots$ by

$$\begin{aligned} x_0 \in D, \quad u_m &= x_m + \varrho H(x_m), \\ y_m &= x_m - [u_m, x_m; H]^{-1} H(x_m), \\ z_m &= y_m - (3I - [u_m, x_m; H]^{-1}([y_m, x_m; H] \\ &\quad + [y_m, u_m; H]))[u_m, x_m; H]^{-1} H(y_m), \\ x_{m+1} &= z_m - [z_m, y_m; H]^{-1}([u_m, x_m; H] + [y_m, x_m; H] \\ &\quad - [z_m, x_m; H])[u_m, x_m; H]^{-1} H(z_m), \end{aligned} \tag{1.2}$$

where $[\cdot, \cdot; H] : D \times D \to \mathcal{L}(B)$ is the first order divided difference of H, $\mathcal{L}(B)$ is the space of bounded bilinear operators from B into B and ϱ is an arbitrary non-zero constant. The seventh order of the method (1.2) is shown provided that $B = \mathbb{R}^k$, where k is a natural number and by assuming the existence of at least sixth derivative and utilizing Taylor series expansions. Hence, the application is limited to solving nonlinear equation (1.1) where the operator is that many times differentiable. But the method may converge even if sixth order derivative $H^{(6)}$ does not exist, where by $H^{(i)}$ we denote the $i-$th derivative of the operator H.

Let us illustrate our concerns with a toy example: Consider D to be an interval containing 0 and 1. Define the function H on D as

$$H(t) = \begin{cases} 7t^3 \log t + 4t^5 - 4t^4, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

It is clear by this definition that $H^{(3)}(t)$ is not continuous at $t = 0$. Thus, the results in [18] cannot assure the convergence of the method to the solution $t^* = 1$. But the method converges if $\varrho = 1$ and $x_0 = 0.95$.

Moreover, notice that there are no derivatives on the method (1.2). Other concerns with local convergence analysis (LCA) provided in [18] are

(C_1) The initial guess x_0 is “shot in dark”.

(C_2) A priori upper bounds on $\|x_m - \delta\|$ are not given, where $\delta \in D$ stands for a solution of the equation (1.1). That is one does not know in advance the number of iterations to be performed to reach a predecided error tolerance.

(C_3) There is no information on the isolation of the solution.

(C_4) There is no assurance that the method converges (although it may converge to δ) if at least $H^{(6)}$ does not exist.

(C_5) The results are limited to the case when $B = \mathbb{R}^k$.

(C_6) The more interesting than LCA semi-local convergence (SLCA) is not given in [18].

(C_7) The same concerns exist for numerous other methods with no derivatives [18, 19].

The present article processes all the concerns positively. In particular, the LCA relies on the general concept on ω-continuity [1, 2, 6, 13] and uses only information from the operators appearing on the method. Moreover, the SLCA utilizes majorizing sequences [1, 13].

The novelty of the article lies in the fact that the process leading to the aforementioned benefits does not rely on the particular method (1.2). But it can be used on other methods utilizing inverses of linear operators in a similar fashion. Notice that the development of efficiency and computational benefits have been discussed in [18]. That is why, we do not repeat these aspects of the method in present article.

The paper is structured as follows: The local convergence in Section 2 is followed by the semilocal convergence in Section 3. The numerical applications and concluding remarks appearing in Section 4 and Section 5, respectively complete the paper.

2 Convergence I: LCA

The LCA relies on the following conditions. Let $T = [0, \infty)$.
Assume:

(A_1) There exists $M \in \mathcal{L}(B)$ such that $M^{-1} \in \mathcal{L}(B)$ exists.
The developed functions "w" are assumed to be symmetric.

(A_2) There exist continuous and nondecreasing functions (CNF) $f : T \to T$, $\omega_0 : T \times T \to T$ such that the equation $\omega_0(f(t), t) - 1 = 0$ has a minimal positive solution denoted by ρ_0. Let $T_0 = [0, \rho_0)$ and $\mathcal{P}(\delta, \rho_0) = \{x \in D : \|x - \delta\| < \rho_0\}$.

(A_3)

$$\|M^{-1}([u, y; H] - M)\| \leq \omega_0(\|u - \delta\|, \|y - \delta\|)$$

and

$$\|u - \delta\| \leq f(\|x - \delta\|)$$

for each $x, y \in D$ and $u = x + \varrho H(x)$.
Let $D_0 = D \cap \mathcal{P}(\delta, \rho_0)$.

(A_4) There exists CNF $\omega : T_0 \times T_0 \to T$, $\omega_1 : T_0 \times T_0 \times T_0 \to T$, $\omega_2 : T_0 \times T_0 \times T_0 \to T$ and $\omega_3 : T_0 \to T$ such that

$$\begin{aligned}
\|M^{-1}([x, y; H] - [z, \delta; H])\| &\leq \omega(\|x - \delta\|, \|y - \delta\|), \\
z = x \quad &\text{or} \quad z = y, \\
\|M^{-1}([u, x; H] - [y, \delta; H])\| &\leq \omega_1(\|x - \delta\|, \|y - \delta\|, \|u - \delta\|) \\
\|M^{-1}([u, x; H] - [y, z; H])\| &\leq \omega_2(\|x - \delta\|, \|y - \delta\|, \|u - \delta\|), \\
z = x \quad &\text{or} \quad z = u
\end{aligned}$$

and

$$\|M^{-1}([x, \delta; H] - M)\| \leq \omega_3(\|x - \delta\|)$$

for each $x, y, u \in D_0$.
The third condition is used to obtain an estimate for $||x_1 - \delta\|$ (2.11)

$$\|M^{-1}([y_0, x_0; H] - [z_0, x_0; H])\| \leq \omega_2(\|x_0 - \delta\|, \|y_0 - \delta\|, \|z_0 - \delta\|).$$

(A_5) The equations $h_1(t) - 1 = 0$, $h_2(t) - 1 = 0$ have minimal positive solutions $s_1, s_2 \in T_0$, where $h_1 : T_0 \to T$, $h_2 : T_0 \to T$ are defined by

$$h_1(t) = \frac{\omega(f(t), t)}{1 - \omega_0(f(t), t)}$$

and

$$h_2(t) = \left[\frac{\omega_1(t, h_1(t)t, f(t))}{1 - \omega_0(f(t), t)} + \frac{2\omega_2(t, h_1(t)t, f(t))(1 + \omega_3(h_1(t)t))}{(1 - \omega_0(f(t), t))^2}\right] h_1(t).$$

(A_6) The equation $\omega_0(h_1(t)t, h_2(t)t) - 1 = 0$ has a minimal positive solution $\overline{\rho} \in T_0$. Let $\rho_1 = \min\{\rho_0, \overline{\rho}\}$ and $T_1 = [0, \rho_1)$. Then

$$\|M^{-1}([x, y; H] - M)\| \leq \overline{\omega_0}(\|x - \delta\|, \|y - \delta\|)$$

for each $x, y \in T_1$, where $\overline{\omega_0}$ is CNF on T_1.

(A_7) The equation $h_3(t) - 1 = 0$ has a minimal positive solution $s_3 \in T_1$, where the function $h_3 : T_1 \to \mathbb{R}$ is defined by

$$h_3(t) = \left[\frac{\omega(h_1(t)t, h_2(t)t)}{1 - \overline{\omega_0}(h_1(t)t, h_2(t)t)} + \frac{\omega_2(t, h_1(t)t, h_2(t)t)(1 + \omega_3(h_2(t)t))}{(1 - \overline{\omega_0}(h_1(t)t, h_2(t)t))(1 - \omega_0(f(t), t))}\right] h_2(t).$$

(A_8) $\mathcal{P}[\delta, s^*] \subset D$, where $s^* = \min\{s_i\}, i = 1, 2, 3$ and $\mathcal{P}[\delta, s^*]$ is the closure of the open ball $\mathcal{P}(\delta, s^*)$.

The parameter s^* given in the condition (A_8) is a convergence radius for the method (1.2) (see Theorem 2.1).

Let $T_2 = [0, s^*)$. It follows that for each $t \in T_2$

$$0 \leq \omega_0(f(t), t) < 1, \tag{2.1}$$

$$0 \leq \omega_0(h_1(t)t, h_2(t)t) < 1 \tag{2.2}$$

and

$$0 \leq h_i(t) < 1. \tag{2.3}$$

The following LCA result relies on the developed terminology and the conditions (A_1)-(A_8).

Theorem 2.1. *Assume the validity of the conditions (A_1)-(A_8). If x_0 is selected from the set $\mathcal{P}(\delta, s^*) - \{\delta\}$ then, the following assertions are validated for the sequence $\{x_m\}$ produced by the method (1.2)*

$$\{x_m\} \subset \mathcal{P}(\delta, s^*), \tag{2.4}$$

$$\|y_m - \delta\| \leq h_1(\|x_m - \delta\|)\|x_m - \delta\| \leq \|x_m - \delta\| < s^*, \tag{2.5}$$

$$\|z_m - \delta\| \leq h_2(\|x_m - \delta\|)\|x_m - \delta\| \leq \|x_m - \delta\| \tag{2.6}$$

and

$$\|x_{m+1} - \delta\| \leq h_3(\|x_m - \delta\|)\|x_m - \delta\| \leq \|x_m - \delta\|, \tag{2.7}$$

provided that the functions h_i are as previously given and the convergence radius s^* is defined in the condition (A_8).

Proof. Induction is employed to show the assertions (2.4)-(2.7). The conditions (A_1)-(A_3) and the estimate (2.1) give

$$\begin{aligned}\|M^{-1}([u_0, x_0; H] - M)\| &\leq \omega_0(\|u_0 - \delta\|, \|x_0 - \delta\|) \\ &\leq \omega_0(f(\|x_0 - \delta\|), \|x_0 - \delta\|) \leq \omega_0(f(s^*), s^*) < 1. \end{aligned}\tag{2.8}$$

The Banach perturbation Lemma involving inverses of linear operators [1, 11, 21] and the estimate (2.8) assure the existence of $[u_0, x_0; H]^{-1}$ and

$$\|[u_0, x_0; H]^{-1}M\| \leq \frac{1}{1 - w_0(f(\|x_0 - \delta\|), \|x_0 - \delta\|)}. \tag{2.9}$$

It also follows that the first iterate y_0 is well defined and we can write

$$\begin{aligned} y_0 - \delta &= x_0 - \delta - [u_0, x_0; H]^{-1}H(x_0) \\ &= [u_0, x_0; H]^{-1}([u_0, x_0; H] - [x_0, \delta; H])(x_0 - \delta). \end{aligned}\tag{2.10}$$

The application of the conditions (A_4)-(A_8), (2.3) (for $i = 1$), (2.9) and (2.10) give in turn

$$\begin{aligned}\|y_0 - \delta\| &\leq \frac{\omega(\|u_0 - \delta\|, \|x_0 - \delta\|)\|x_0 - \delta\|}{1 - w_0(f(\|x_0 - \delta\|), \|x_0 - \delta\|)} \\ &\leq h_1(\|x_0 - \delta\|)\|x_0 - \delta\| \leq \|x_0 - \delta\| < s^*,\end{aligned}$$

thus, the assertion (2.5) is validated if $m = 0$ and the iterate $y_0 \in \mathcal{P}(\delta, s^*)$. The second iterate z_0 is also well defined by (2.9) and we can similarly write in turn that

$$\begin{aligned} z_0 - \delta = & y_0 - \delta - [u_0, x_0; H]^{-1} H(y_0) \\ & + [u_0, x_0; H]^{-1}\Big(([u_0, x_0; H] - [y_0, x_0; H]) \\ & + ([u_0, x_0; H] - [y_0, u_0; H]) \Big) [u_0, x_0; H]^{-1} H(y_0) \end{aligned}$$

leading to

$$\begin{aligned} \|z_0 - \delta\| \leq & \Big[\frac{\omega_1(\|x_0 - \delta\|, \|y_0 - \delta\|, \|u_0 - \delta\|)}{1 - w_0(f(\|x_0 - \delta\|), \|x_0 - \delta\|)} \\ & + \frac{2\omega_2(\|x_0 - \delta\|, \|y_0 - \delta\|, \|u_0 - \delta\|)(1 + \omega_3(\|y_0 - \delta\|))}{(1 - \omega_0(f(\|x_0 - \delta\|), \|x_0 - \delta\|))^2} \Big] \|y_0 - \delta\| \\ \leq & h_2(\|x_0 - \delta\|)\|x_0 - \delta\| \leq \|x_0 - \delta\|, \end{aligned}$$

where we also used

$$\begin{aligned} \|M^{-1} H(y_0)\| &= \|M^{-1}(H(y_0) - H(\delta))\| = \|M^{-1}[y_0, \delta; H](y_0 - \delta)\| \\ &= \|M^{-1}([y_0, \delta; H] - M + M)(y_0 - \delta)\| \\ &\leq [1 + \|M^{-1}([y_0, \delta; H] - M)\|]\|y_0 - \delta\| \\ &\leq (1 + \omega_3(\|y_0 - \delta\|))\|y_0 - \delta\| \end{aligned}$$

so the assertion (2.6) is validated if $m = 0$ and the iterate $z_0 \in \mathcal{P}(\delta, s^*)$.
The conditions (A_1)-(A_6) imply

$$\begin{aligned} \|M^{-1}([z_0, y_0; H] - M)\| \leq & \overline{\omega_0}(\|z_0 - \delta\|, \|y_0 - \delta\|) \\ \leq & \overline{\omega_0}(h_2(\|x_0 - \delta\|)\|x_0 - \delta\|, h_1(\|x_0 - \delta\|)\|x_0 - \delta\|) \\ \leq & \overline{\omega_0}(h_2(s^*)s^*, h_1(s^*)s^*) < 1, \end{aligned}$$

thus $[z_0, y_0; H]^{-1} \in \mathcal{L}(B)$,

$$\|[z_0, y_0; H]^{-1} M\| \leq \frac{1}{1 - \overline{w_0}(h_2(\|x_0 - \delta\|)\|x_0 - \delta\|, h_1(\|x_0 - \delta\|)\|x_0 - \delta\|)}$$

and the third iteration is also well defined.
Next, we can write in turn

$$\begin{aligned} x_1 - \delta = & z_0 - \delta - [z_0, y_0; H]^{-1} H(z_0) \\ & - [z_0, y_0; H]^{-1}([y_0, x_0; H] - [z_0, x_0; H])[u_0, x_0; H]^{-1} H(z_0) \\ = & [z_0, y_0; H]^{-1}([z_0, y_0; H] - [z_0, \delta; H])(z_0 - \delta) \\ & - [z_0, y_0; H]^{-1}([y_0, x_0; H] - [z_0, x_0; H])[u_0, x_0; H]^{-1} H(z_0) \end{aligned}$$

leading to

$$\begin{aligned} \|x_1 - \delta\| \leq & \Big[\frac{\omega(\|y_0 - \delta\|, \|z_0 - \delta\|)}{1 - \overline{w_0}(\|z_0 - \delta\|, \|y_0 - \delta\|)} \\ & + \frac{\omega_2(\|x_0 - \delta\|, \|y_0 - \delta\|, \|z_0 - \delta\|)(1 + \omega_3(\|z_0 - \delta\|))}{(1 - \overline{\omega_0}(\|z_0 - \delta\|, \|y_0 - \delta\|))(1 - \omega_0(f(\|x_0 - \delta\|), \|x_0 - \delta\|))} \Big] \|z_0 - \delta\| \\ \leq & h_3(\|x_0 - \delta\|)\|x_0 - \delta\| \leq \|x_0 - \delta\|, \end{aligned} \tag{2.11}$$

therefore the assertions (2.4) (for $m = 1$) and (2.7) are validated if $m = 0$ and the iterate $x_1 \in \mathcal{P}(\delta, s^*)$, where we used

$$\begin{aligned}\|M^{-1}H(z_0)\| &= \|M^{-1}(H(z_0) - H(\delta))(z_0 - \delta)\| \\ &= \|M^{-1}([z_0, \delta; H] - M + M)(z_0 - \delta)\| \\ &\le [1 + \|M^{-1}([z_0, \delta; H] - M)\|]\|z_0 - \delta\| \\ &\le (1 + \omega_3(\|z_0 - x^*\|))\|z_0 - x^*\|.\end{aligned}$$

The induction is validated for the assertions (2.5)-(2.7) for $m = 0$ and for $m = 1$ in the assertion (2.4). Simply exchange the iterates x_0, y_0, z_0, x_1, respectively by x_m, y_m, z_m, x_{m+1} to terminate the induction for all $m = 0, 1, 2, \ldots$. Then, it follows from the estimation

$$\|x_{m+1} - \delta\| \le d\|x_m - \delta\| < s^*,$$

where $d = h_3(\|x_0 - x^*\|) \in [0, 1)$ that the iterate $x_{m+1} \in \mathcal{P}(\delta, s^*)$ and $\lim_{m\to+\infty} x_m = \delta$. □

The functions "ω" and "f" are left uncluttered in Theorem 2.1.

Remark 2.2. *A possible choice of the function f is motivated by the calculations:*

$$\begin{aligned}u - \delta &= x - \delta + \varrho H(x) = (I + \varrho[x, \delta; H])(x - \delta) \\ &= [I + \varrho M + \varrho M M^{-1}([x, \delta; H] - M)](x - \delta),\end{aligned}$$

so

$$\|u - \delta\| \le [\|I + \varrho M\| + |\varrho|\|M\|\omega_3(\|x - \delta\|)]\,\|x - \delta\|.$$

Hence, we can choose

$$f(t) = [\|I + \varrho M\| + |\varrho|\|M\|\omega_3(t)]\,t.$$

Assume:

$$\begin{aligned}\|M^{-1}([u, y; H] - M)\| &\le c_0\|u - \delta\| + d_0\|y - \delta\| + d_1, \\ \|M^{-1}([x, \delta; H] - M)\| &\le c_1\|x - \delta\| + d_2, \\ \|M^{-1}([u, x; H] - [x, \delta; H])\| &\le c_2\|u - \delta\| + c_3\|x - \delta\| + d_3,\end{aligned}$$

by using the first conditions in (A_4) for $z = x$.

$$\|M^{-1}([u, x; H] - [y, \delta; H])\| \le c_4\|u - \delta\| + c_5\|x - \delta\| + c_6\|y - \delta\| + d_4$$

and

$$\|M^{-1}([u, x; H] - [y, z; H])\| \le c_7\|x - \delta\| + c_8\|y - \delta\| + c_9\|u - \delta\| + d_5$$

hold instead. Then, we can choose

$$\begin{aligned}\omega_0(s, t_1) &= c_0 s + d_0 t_1 + d_1, \\ \omega_3(t) &= c_1 t + d_2, \\ \omega(s, t) &= c_2 s + c_3 t + d_3, \\ \omega_1(s, t, t_1) &= c_4 s + c_5 t + c_6 t_1 + d_4\end{aligned}$$

and

$$\omega_2(s, t, t_1) = c_7 s + c_8 t + c_9 t_1 + d_5,$$

where the coefficients "c" and "d" are nonnegative parameters.
The condition (A_8) can be replaced by

$(A_8)'$ $\mathcal{P}(\delta, \bar{s}) \subset D$,

where $\bar{s} = \max\{s^*, f(s^*)\}$. *In this case condition* $\|u - \delta\| \leq f(\|x - \delta\|)$ *can be dropped.*
Some possible choices for the linear operator M *are:*
The differentiable case: $M = H'(\delta)$.
The non differentiable case: $M = [u_0, x_0; H]$ *provided that* $[u_0, x_0; H]^{-1} \in \mathcal{L}(B)$.
Other choices should be taking into account optimization concerns.

The isolation of the solution domain is specified in the next result.

Proposition 2.3. *Assume:*
There exists a solution $\bar{x} \in \mathcal{P}(\delta, \rho_2)$ *of the equation* $H(x) = 0$ *for some* $\rho_2 > 0$*; The condition* (A_1) *and the first condition in* (A_2) *are validated on the ball* $\mathcal{P}(\delta, \rho_2)$ *and there exists* $\rho_3 \geq \rho_2$ *such that*

$$\omega_3(\rho_3) < 1.$$

Then, the equation $H(x) = 0$ *is uniquely solvable by* δ *in the domain* $D_3 = D \cap \mathcal{P}[\delta, \rho_3]$.

Proof. Define the divided difference $[\bar{x}, \delta; H]$. Then, we get

$$\begin{aligned}\|M^{-1}([\bar{x}, \delta; H] - M)\| \leq& \omega_3(\|\bar{x} - \delta\|)\\ \leq& \omega_3(\rho_3) < 1,\end{aligned}$$

thus

$$\begin{aligned}\bar{x} - \delta =& [\bar{x}, \delta; H]^{-1}(H(\bar{x}) - H(\delta))\\ =& [\bar{x}, \delta; H]^{-1}(0) = 0.\end{aligned}$$

Hence, we conclude $\bar{x} = \delta$. □

A possible choice for $\rho_2 = s^*$.

3 Convergence II: SLCA

In this Section the role of δ and the functions "ω" is replaced by x_0 and the "v" functions. However, similar calculations take place.
Assume:

(Γ_1) = (A_1).
The developed functions "v" are assumed to be symmetric.

(Γ_2) There exists CNF $g : T \to T$, $v_0 : T \times T \to T$ such that the equation $v_0(g(t), t) - 1 = 0$ has a minimal positive solution $\rho_3 \in T$.

(Γ_3)

$$\begin{aligned}\|M^{-1}([u, y; H] - M)\| \leq v_0(\|u - x_0\|, \|y - x_0\|) \quad \text{for each} \quad x, y \in D,\\ u = x + \varrho H(x),\\ \|u - x_0\| \leq g(\|x - x_0\|),\end{aligned}$$

and

$$v_0(g(\|\varrho H(x_0)\|), 0) < 1$$

Let $T_3 = [0, \rho_3)$.

(Γ_4) The equation $v_0(t, t) - 1 = 0$ has a minimal positive solution $\tilde{\rho} \in T_3$. Let $\rho_4 = \min\{\rho_3, \tilde{\rho}\}$, $T_4 = [0, \rho_4)$ and $D_4 = D \cap \mathcal{P}(x_0, \rho_4)$. Then

$$\|M^{-1}([x, y; H] - M)\| \leq \overline{v_0}(\|x - x_0\|, \|y - x_0\|)$$
$$\text{for each} \quad x, y \in D_4, \quad \text{where} \quad \overline{v_0} \quad \text{is CNF on } T_4.$$

(Γ_5) There exists a CNF $v: T_4 \times T_4 \times T_4 \to T$ such that

$$\|M^{-1}([u, y; H] - [z, x; H])\| \leq v(\|x - x_0\|, \|u - x_0\|, \|z - x_0\|),$$

for each $x, y, z, u \in D_4$.
The condition is used to obtain the estimates

$$\begin{aligned}
\|M^{-1}([u_m, x_m; H] - [y_m, x_m; H])\| &= \|M^{-1}([y_m, x_m; H] - [u_m, x_m; H])\| \\
&\leq v(\|x_m - x_0\|, \|y_m - x_0\|, \|u_m - x_0\|), \\
\|M^{-1}([u_m, x_m; H] - [y_m, u_m; H])\| &= \|M^{-1}([y_m, u_m; H] - [u_m, x_m; H])\| \\
&\leq v(\|x_m - x_0\|, \|y_m - x_0\|, \|u_m - x_0\|), \\
\|M^{-1}([y_m, x_m; H] - [z_m, x_m; H])\| &= \|M^{-1}([z_m, x_m; H] - [y_m, x_m; H]) \\
&\| \leq v(\|x_m - x_0\|, \|y_m - x_0\|, \|z_m - x_0\|).
\end{aligned}$$

(Γ_6) Define the real sequence $\{a_n\}$ for $a_0 = 0$, some $b_0 \geq 0$ as

$$\begin{aligned}
c_m =& b_m + \left[\frac{v(a_m, b_m, g(a_m))}{1 - v_0(g(a_m), a_m)} + \frac{2v(a_m, b_m, g(a_m))^2}{(1 - v_0(g(a_m), a_m))^2}\right](b_m - a_m), \qquad (3.1)\\
\delta_m =& (1 + \overline{v_0}(b_m, c_m))(c_m - b_m) + v(a_m, b_m, g(a_m))(b_m - a_m), \\
a_{m+1} =& c_m + \frac{1}{1 - \overline{v_0}(c_m, b_m)}\left[1 + \frac{v(a_m, b_m, c_m)}{1 - v_0(g(a_m), a_m)}\right]\delta_m, \\
\lambda_{m+1} =& (1 + \overline{v_0}(a_{m+1}, a_m))(a_{m+1} - a_m) \\
&+ (1 + v_0(g(a_m), a_m))(b_m - a_m),
\end{aligned}$$

and

$$b_{m+1} = a_{m+1} + \frac{\lambda_{m+1}}{1 - v_0(g(a_{m+1}), a_{m+1})}.$$

There exists $\rho_5 \in (0, \rho_4)$ such that

$$v_0(g(a_m), a_m) < 1, \overline{v_0}(c_m, b_m) < 1 \quad \text{and} \quad a_m < \rho_5.$$

Notice that the sequence $\{a_m\}$ is nondecreasing by the formula (3.1) and the preceding condition. It follows $a^* = \lim_{m \to +\infty} a_m$ exists.

(Γ_7) $\mathcal{P}[x_0, a^*] \subset D$.

(Γ_8) Notice that $[u_0, x_0; H]^{-1} \in \mathcal{L}(B)$ by the conditions (Γ_1)-(Γ_3). Choose b_0 such that $\|[u_0, x_0; H]^{-1} H(x_0)\| \leq b_0$.

Then, the SCLA result for the method (1.2) is:

Theorem 3.1. *Assume that the conditions* $(\Gamma_1) - (\Gamma_7)$ *are validated. Then, the following assertions are validated for the sequence* $\{x_m\}$ *given by method (1.2)*

$$\{x_m\} \subset \mathcal{P}(x_0, a^*), \qquad (3.2)$$

$$\|y_m - x_m\| \leq b_m - a_m, \qquad (3.3)$$

$$\|z_m - y_m\| \leq c_m - b_m, \qquad (3.4)$$

$$\|x_{m+1} - z_m\| \leq a_{m+1} - c_m \qquad (3.5)$$

and there exists $\delta \in \mathcal{P}[x_0, a^*]$ *satisfying the equation* $H(x) = 0$*, so that*

$$\|\delta - x_m\| \leq a^* - a_m. \qquad (3.6)$$

Proof. As in the local case induction is employed to verify the assertions (3.2)-(3.5). The formula (3.1), the method (1.2) and the condition (Γ_8) imply

$$\|y_0 - x_0\| = \|[u_0, x_0; H]^{-1} H(x_0)\| \leq b_0 = b_0 - a_0 < a^*,$$

thus the iterate $y_0 \in \mathcal{P}(x_0, a^*)$ and the assertion (3.3) is validated if $m = 0$. Then, by simply following the calculations of the local case and exchanging δ, A by x_0, Γ conditions, respectively we obtain the estimates

$$\begin{aligned}
\|[u_m, x_m; H]^{-1} M\| &\leq \frac{1}{1 - v_0(g(\|x_m - x_0\|), \|x_m - x_0\|)},\\
z_m - y_m &= -[u_m, x_m; H]^{-1} H(y_m) - [u_m, x_m; H]^{-1}\\
&\quad \times \Big(([u_m, x_m; H] - [y_m, x_m; H]) + ([u_m, x_m; H] - [y_m, u_m; H]) \Big)\\
&\quad \times [u_m, x_m; H]^{-1} ([y_m, x_m; H] - [u_m, x_m; H])(y_m - x_m),
\end{aligned}$$

since by the first substep

$$\begin{aligned}
H(y_m) &= H(y_m) - H(x_m) - [u_m, x_m; H](y_m - x_m)\\
&= ([y_m, x_m; H] - [u_m, x_m; H])(y_m - x_m)
\end{aligned}$$

leading to

$$\begin{aligned}
\|z_m - y_m\| \leq & \Bigg[\frac{v(\|x_m - x_0\|, \|y_m - x_0\|, \|u_m - x_0\|)}{1 - v_0(g(\|x_m - x_0\|), \|x_m - x_0\|)}\\
& + \frac{2v(\|x_m - x_0\|, \|y_m - x_0\|, \|u_m - x_0\|)^2}{(1 - v_0(g(\|x_m - x_0\|), \|x_m - x_0\|))^2} \Bigg] \|y_m - x_m\|\\
\leq & \frac{v(a_m, b_m, g(a_m))}{1 - v_0(g(a_m), a_m)} \left[1 + \frac{2v(a_m, b_m, g(a_m))}{1 - v_0(g(a_m), a_m)} \right] (b_m - a_m)\\
= & c_m - b_m,
\end{aligned}$$

$$\begin{aligned}
\|z_m - x_0\| &\leq \|z_m - y_m\| + \|y_m - x_0\|\\
&\leq c_m - b_m + b_m - a_0 = c_m < a^*.
\end{aligned}$$

Thus, the iterate $z_m \in \mathcal{P}(x_0, a^*)$ and the assertion (3.4) is validated. Moreover,

$$\|[z_m, y_m; H]^{-1} M\| \leq \frac{1}{1 - \overline{v_0}(\|z_m - x_0\|, \|y_m - x_0\|)},$$

x_{m+1} exists,

$$\begin{aligned}
x_{m+1} - z_m = & -[z_m, y_m; H]^{-1} H(z_m)\\
& - [z_m, y_m; H]^{-1} ([y_m, x_m; H] - [z_m, x_m; H])[u_m, x_m; H]^{-1} H(z_m),
\end{aligned}$$

leading to

$$\begin{aligned}
\|x_{m+1} - z_m\| \leq & \Bigg[\frac{1}{1 - \overline{v_0}(\|z_m - x_0\|, \|y_m - x_0\|)}\\
& + \frac{v(\|x_m - x_0\|, \|y_m - x_0\|, \|z_m - x_0\|)}{(1 - \overline{v_0}(\|z_m - x_0\|, \|y_m - x_0\|))(1 - v_0(\|u_m - x_0\|, \|x_m - x_0\|))} \Bigg] \|M^{-1} H(z_m)\|. \qquad (3.7)
\end{aligned}$$

But

$$\begin{aligned}H(z_m) &= H(z_m) - H(y_m) + H(y_m)\\ &= [z_m, y_m; H](z_m - y_m) + ([y_m, x_m; H] - [u_m, x_m; H])(y_m - x_m),\end{aligned}$$

so

$$\begin{aligned}\|M^{-1}H(z_m)\| \leq &(1 + \overline{v_0}(\|z_m - x_0\|, \|y_m - x_0\|))\|(z_m - y_m)\\ &+v(\|x_m - x_0\|, \|y_m - x_0\|, \|u_m - x_0\|)\|(y_m - x_m)\| = \overline{\delta_m} \leq \delta_m. \end{aligned}\tag{3.8}$$

Then, (3.7) and (3.8) give

$$\begin{aligned}\|x_{m+1} - z_m\| &\leq \frac{1}{1 - \overline{v_0}(c_m, b_m)}\left[1 + \frac{v(a_m, b_m, c_m)}{1 - v_0(g(a_m), a_m)}\right]\delta_m\\ &= a_{m+1} - c_m\end{aligned}$$

and

$$\begin{aligned}\|x_{m+1} - x_0\| &\leq \|x_{m+1} - z_m\| + \|z_m - x_0\| \leq a_{m+1} - c_m + c_m - a_0\\ &= a_{m+1} < a^*.\end{aligned}$$

Thus, the iterate $x_{m+1} \in \mathcal{P}(x_0, a^*)$ and the assertions (3.3) and (3.5) are validated. The induction is terminated if the assertion (3.4) is validated for m replacing $m+1$.
We can write by the first substep in turn that

$$\begin{aligned}H(x_{m+1}) =&H(x_{m+1}) - H(x_m) - [u_m, x_m; H](y_m - x_m)\\ =&[x_{m+1}, x_m; H](x_{m+1} - x_m) - [u_m, x_m; H](y_m - x_m),\end{aligned}$$

leading to

$$\begin{aligned}\|M^{-1}H(x_{m+1})\| \leq &(1 + \overline{v_0}(\|x_{m+1} - x_0\|, \|x_m - x_0\|))\|(x_{m+1} - x_m)\\ &+(1 + \overline{v_0}(\|u_m - x_0\|, \|x_m - x_0\|))\|(y_m - x_m)\| = \overline{\lambda}_{m+1}\\ &\leq \lambda_{m+1},\end{aligned}\tag{3.9}$$

therefore,

$$\begin{aligned}\|y_{m+1} - x_{m+1}\| &\leq \|[u_{m+1}, x_{m+1}; H]^{-1}M\|\|M^{-1}H(x_{m+1})\|\\ &\leq \frac{\overline{\lambda}_{m+1}}{1 - v_0(\|u_{m+1} - x_0\|, \|x_{m+1} - x_0\|)}\\ &\leq \frac{\lambda_{m+1}}{1 - v_0(g(a_{m+1}), a_{m+1})} = b_{m+1} - a_{m+1}\end{aligned}$$

and

$$\begin{aligned}\|y_{m+1} - x_0\| &\leq \|y_{m+1} - x_{m+1}\| + \|x_{m+1} - x_0\| \leq b_{m+1} - a_{m+1} + a_{m+1} - a_0\\ &= b_{m+1} < a^*.\end{aligned}$$

Hence, the induction for the assertions (3.2)-(3.6) is terminated and the iterates $x_m, y_m, z_m \in \mathcal{P}(x_0, a^*)$. It follows from the validity of these assertions and the convergence of the real sequence $\{a_m\}$ that the sequence $\{x_m\}$ is Cauchy. But B is a Banach space. Therefore, there exists $\delta \in \mathcal{P}[x_0, a^*]$ so that $\lim_{m\to+\infty} x_m = \delta$. Moreover, the estimate (3.9) for $m \to +\infty$ and the continuity of the operator H imply $H(\delta) = 0$. Furthermore, the estimate

$$\|x_{m+j} - x_m\| \leq a_{m+j} - a_m$$

becomes (3.6) if $j \to +\infty$. □

Remark 3.2. *Assume:*

$(\Gamma_3)'$ $\|M^{-1}([x,x_0;H]-M)\| \leq v_1(\|x-x_0\|)$

for each $x \in D$, where $v_1 : T \to \mathbb{R}$ is CNF.

The condition is weaker than the first condition in (Γ_3) and the function v_1 is tighter than v_0. Then, the motivation for the definition of the function g follows from the calculation

$$\begin{aligned} u - x_0 &= x - x_0 + \varrho H(x) - \varrho H(x_0) + \varrho H(x_0) \\ &= (I + \varrho[x, x_0; H])(x - x_0) + \varrho H(x_0), \end{aligned}$$

so

$$\|u - x_0\| = (\|I + \varrho M\| + |\varrho|\|M\|v_1(\|x - x_0\|))\|x - x_0\| + |\varrho|\|H(x_0)\|.$$

Hence, we can choose

$$g(t) = (\|I + \varrho M\| + |\varrho|\|M\|v_1(t))t + |\varrho|\|H(x_0)\|.$$

Under this choice of g the second condition in (Γ_3) can be dropped.

The condition (Γ_7) can be replaced by

$(\Gamma_7)'$ *$\mathcal{P}[x_0, \overline{a}] \subset D$, where $\overline{a} = \max\{a^*, g(a^*)\}$.*

The function v_0, v can be chosen as the ones in Remark 2.2 with x_0 replacing δ.

Possible choices for the linear operator M are:

The differentiable case: $M = H'(x_0)$.

The non differentiable case: $M = [u_0, x_0; H]$ provided that $[u_0, x_0; H]^{-1} \in \mathcal{L}(B)$.

Other choices are chosen depending on the concrete application at hand so the convergence results are optimized.

The isolation of a solution region is determined next.

Proposition 3.3. *Assume :*

There exists a solution $\overline{x} \in \mathcal{P}(x_0, \rho_5)$ of solution $H(x) = 0$ for some $\rho_5 > 0$, The first condition in (Γ_3) is validated on the ball $\mathcal{P}(x_0, \rho_5)$ and there exists $\rho_6 \geq \rho_5$ such that

$$\begin{aligned} \|M^{-1}([\overline{x}, x; H] - M)\| \leq& v_2(\|\overline{x} - x_0\|, \|x - x_0\|), \\ v_2(\rho_5, \rho_6) <& 1, \end{aligned}$$

where $v_2 : T \times T \to \mathbb{R}$ is CNF.

Let $D_4 = D \cap \mathcal{P}[x_0, \rho_6]$. Then, the only solution of the equation $H(x) = 0$ is $\overline{x}$ in the region D_4.

Proof. Let $\overline{y} \in D_4$ satisfy $H(\overline{y}) = 0$. Define the divided difference $[\overline{x}, \overline{y}; H]$. This is possible if $\overline{y} \neq \overline{x}$. Then, we obtain in turn by the conditions

$$\begin{aligned} \|M^{-1}([\overline{x}, \overline{y}; H] - M)\| \leq& v_2(\|\overline{x} - x_0\|, \|\overline{y} - x_0\|) \\ \leq& v_2(\rho_5, \rho_6) < 1, \end{aligned}$$

thus $[\overline{x}, \overline{y}; H]^{-1} \in \mathcal{L}(B)$. But the identity

$$0 = H(\overline{x}) - H(\overline{y}) = [\overline{x}, \overline{y}; H](\overline{x} - \overline{y})$$

leads to a contradiction and the divided difference $[\overline{x}, \overline{y}; H]$ can not be defined. Therefore, we conclude that $\overline{y} = \overline{x}$. □

Remark 3.4.

(1) *The condition (Γ_7) can also be replaced by*

$(\Gamma_7)''$ *$\mathcal{P}(x_0, \rho_2) \subset D$.*

(2) *Under all the assumptions (Γ_1)-(Γ_8) in Proposition 3.3, set $\overline{x} = \delta$ and $a^* = \rho_5$.*

4 Numerical examples

A local example is given first. The solution in the second example is obtained by using the method (1.2) to solve a non differentiable equation.

Example 4.1. Let $B = \mathbb{R}^3$ and $D = \mathcal{P}[\delta, 1]$. Consider the mapping on the ball D be given for $\gamma = (\alpha_1, \alpha_2, \alpha_3)^\top$ as

$$H(\gamma) = \left(\frac{e-1}{2}\alpha_1^2 + \alpha_1 + p_1\alpha_2^2 + p_2\alpha_3^2, \alpha_2 + p_3\alpha_1^2 + p_4\alpha_3^2, p_5\alpha_1^2 + p_6\alpha_2^2 + e^{\alpha_3} - 1\right)^\top,$$

where $p_i, i = 1, 2, \ldots, 6$ are real parameters.
The Jacobian is given by

$$H'(\gamma) = \begin{pmatrix} (e-1)\alpha_1 + 1 & 2p_1\alpha_2 & 2p_1\alpha_3 \\ 2p_3\alpha_1 & 1 & 2p_4\alpha_3 \\ 2p_5\alpha_1 & 2p_6\alpha_2 & e^{\alpha_3} \end{pmatrix}.$$

It follows that the solution $\gamma^* = (0, 0, 0)^\top$ and $H'(\gamma^*) = I$ the identity mapping with choice $M = H'(\gamma^*)$. Then, the conditions (A_1)-(A_7) are validated with f as given in Remark 2.2 if we choose for
$[x, y; H] = \int_0^1 H'(y + \theta(x-y))d\theta$:

$$\begin{aligned} \omega_0(s,t) &= \frac{1}{2}(e-1)(s+t), \\ \omega(s,t) &= \frac{1}{2}(e-1)(2t+s), \\ \omega_1(t_1,t_2,t_3) &= \frac{1}{2}(e-1)(t_1+t_2+t_3), \\ \omega_2(t_1,t_2,t_3) &= \omega_1(t_1,t_2,t_3) \end{aligned}$$

and

$$\omega_3(t) = \frac{1}{2}(e-1)t.$$

Taking $\varrho = 1$ and using condition (A_8), the radius of convergence $s^* = 0.10828402883335114$.

Example 4.2. Let $H : B \to B$ be a mapping. Recall that the standard divided difference of order one when $B = \mathbb{R}^k$ is defined for $\bar{l} = (l_1, l_2, \ldots, l_k)^\top$, $\bar{q} = (q_1, q_2, \ldots, q_k)^\top$, $i = 1, 2, 3, \ldots, k$, $j = 1, 2, 3, \ldots, k$ by

$$[\bar{q}, \bar{l}; H]_{ji} = \frac{H_j(q_1, \ldots, q_{i-1}, q_i, l_{i+1}, \ldots, l_k) - H_j(q_1, \ldots, q_{i-1}, l_i, l_{i+1}, \ldots, l_k)}{q_i - l_i}$$

provided that $q_i \neq l_i$.
The solution is sought for the nonlinear system

$$\begin{cases} 3t_1^2t_2 + t_2^2 - 1 + |t_1 - 1| &= 0, \\ t_1^4 + t_1t_2^3 - 1 + |t_2| &= 0. \end{cases}$$

Let $H = (H_1, H_2)$ for $(t_1, t_2) \in \mathbb{R}^2$, where

$$\begin{aligned} H_1(t_1,t_2) =& 3t_1^2t_2 + t_2^2 - 1 + |t_1 - 1| \quad \text{and} \\ H_2(t_1,t_2) =& t_1^4 + t_1t_2^3 - 1 + |t_2|. \end{aligned}$$

Then, the system becomes

$$H(t) = 0 \quad \text{for} \quad t = (t_1, t_2)^\top.$$

The divided difference $L = [\cdot, \cdot; H]$ belongs in the space $M_{2\times 2}(\mathbb{R})$ and is the standard 2×2 matrix in $\mathbb{R}^2$ [7]. Let us choose $x_0 = (5, 5)^\top$. Then, the application of the method (1.2) gives the solution t^* after three iterations. The solution $t^* = (t_1^*, t_2^*)^\top$, where

$$t_1^* = 0.894655373334687$$

and

$$t_2^* = 0.327826421746298.$$

5 Conclusion

A number of problems arising in diverse disciplines are formulated in form of an equation or system of equations in abstract spaces many of which are nondifferentiable. Thus, to handle the solutions of such equations, researchers have developed the methodology to overcome this issue which can also be applied to solve the differentiable equations. However, the same methodology can be applied on other methods utilizing the inverses of linear operators. Further, numerical experiments are performed that demonstrates the theoretical part. In our future research the methodology shall be extended to extend the applicability of multipoint and multi-step methods [1,9,11,14–16,18,19,21,22].

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