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A numerical method without accuracy saturation for a fractional-order evolution equation in a Banach space

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Анотація. Ми розглядаємо задачу Коші для еволюційного рівняння дробового порядку з сильно додатним оператором у банаховому просторі. Розроблено та обґрунтовано чисельний метод, що не залежить від ефекту насичення точності, як для однорідного, так і для неоднорідного випадку. Запропонований підхід ґрунтується на застосуванні перетворення Келі до операторного коефіцієнта та апроксимації оператора розв'язку частковою сумою ряду, що його визначає. Детальний аналіз показує, що метод забезпечує збіжність степеневого типу, яка залежить від гладкості вхідних даних.

Abstract. We study an initial-value problem for a fractional-order evolution equation involving a strongly positive operator in a Banach space. A numerical method free from accuracy saturation is developed and justified for both the homogeneous and inhomogeneous cases. The approach is based on applying the Cayley transform to the operator coefficient and approximating the solution operator by a partial sum of its defining series. A detailed analysis reveals a power-type convergence rate that depends automatically on the smoothness of the input data.

1 Introduction

In contemporary science and engineering, there is growing interest in modeling complex phenomena that exhibit memory effects, fractal behavior, nonlocal interactions, and anomalous or nonlinear dynamics. Classical modeling techniques often fall short of capturing these features with sufficient generality. In contrast, fractional integro-differential calculus offers a more flexible and accurate framework for representing such intricate behaviors in real-world systems. As a result, fractional models have found increasing relevance across diverse domains, including materials science, biology, finance, signal processing, and many others. These topics have been extensively addressed in a wide range of scholarly monographs (see, for example, [1–8]), several of which have attained the status of classical references in the field.

Since closed-form solutions are typically elusive for such equations, the practical adoption of fractional calculus has been significantly facilitated by advances in numerical methods and computational tools, which have made it feasible to employ these models in both theoretical research and applied studies. Just to mention a few, [9] provides a comprehensive and systematic treatment of finite-difference schemes tailored specifically for fractional differential equations with detailed accuracy and convergence analysis. It thoroughly addresses time-fractional sub-diffusion and wave

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equations, space-fractional PDEs, time-space distributed-order fractional models and other related topics. The monograph [10] presents novel fractal-fractional operators, numerical methods with stability and error analysis, and applications to chaotic and epidemic models. The monograph [11] delivers a detailed and mathematically rigorous framework for solving time-fractional evolution equations, combining finite element spatial discretization, robust time-stepping schemes, and complete stability and error analysis, making it valuable for both theoretical exploration and computational implementation.

The article [12] also contributes to this field by rigorously addressing Dirichlet boundary value problems for differential equations involving Riemann–Liouville fractional derivatives. By reformulating the problem into an equivalent integral equation, detailed regularity properties of the exact solution are established. Subsequently, a numerical grid method is developed, and weighted error estimates are derived that explicitly capture the influence of boundary conditions, thereby demonstrating enhanced accuracy near the boundary compared to the interior of the domain.

In the present paper, we study the Cauchy problem for an evolution equation formulated within the framework of [13]

$$\begin{gathered} \partial_t u + \partial_t^{-\alpha} A u = f(t), \quad t \in (0;T], \\ u(0) = u_0, \end{gathered} \tag{1.1}$$

where A is a strongly positive operator in a Banach space E, $f : [0;T] \to E$ is a known vector-valued function, $u : [0;T] \to E$ is an unknown solution, $\partial_t = d/dt$, and

$$(\partial_t^{-\alpha} u)(t) = \begin{cases} \partial_t \displaystyle\int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} u(s)\, ds & \text{if } \ -1 < \alpha \leqslant 0; \\ \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\, ds & \text{if } \ 0 < \alpha < 1. \end{cases}$$

In many applications, A is taken to be a linear second-order differential operator involving partial derivatives with respect to spatial variables and time-independent coefficients.

To solve problem (1.1), McLean and Thomée present in [13] a numerical method based on Laplace transform techniques. A key component of their approach is the numerical inversion of the Laplace transform using contour integral representations combined with quadrature rules, particularly trapezoidal-type quadrature along suitably chosen complex contours. This enables accurate and efficient evaluation of solutions at discrete time points, with convergence analysis and numerical examples illustrating the method's effectiveness for problems with memory effects. In particular, the case $-1 < \alpha < 0$ corresponds to the modeling of a subdiffusion process, where the mean square displacement of the diffusing particles is proportional to $t^{1+\alpha}$. For $\alpha = 0$, equation (1.1) reduces to the classical parabolic equation, where the mean square displacement grows linearly with time, i.e., is proportional to t. Meanwhile, the case $0 < \alpha < 1$ is of particular relevance for modeling viscoelastic behaviour.

Unlike [13], the present paper applies the Cayley transform approach. Some insight into this technique is provided, for instance, in [14] which develops a Cayley transform-based method for boundary value problems in a Banach space and establishes weighted error estimates for truncated-series approximations, with convergence rates depending on the smoothness of the input data.

The remaining sections of the present paper are organized as follows. Section 2 provides preliminary material. Section 3, which forms the main part of the paper, is devoted to the analysis of the Cayley transform method for both homogeneous and inhomogeneous equations. Section 4 discusses numerical applications that make use of the convergence results established in the previous section. Finally, Section 5 concludes the paper with some closing remarks.

2 Preliminaries

We recall some definitions, concepts, and formulas needed later.

Definition 2.1. *Let E be a Banach space. A closed linear operator $A : D(A) \to E$ with dense domain $D(A) \subset E$ and resolvent set $\rho(A) \subset \mathbb{C}$ is called strongly positive [15] if there exist constants $\varphi \in (0; \pi/2)$, $\gamma > 0$, and $L > 0$ such that*

$$\left\|(zI - A)^{-1}\right\| \leqslant \frac{L}{1+|z|} \quad \text{for all } z \in \Sigma, \tag{2.1}$$

where I denotes the identity operator, and

$$\Sigma = \left\{z \in \mathbb{C} \mid \varphi \leqslant |\arg z| \leqslant \pi\right\} \cup \left\{z \in \mathbb{C} \mid |z| \leqslant \gamma\right\} \subset \rho(A).$$

By means of the Mittag-Leffler function [16]

$$E_\mu(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+j\mu)}, \quad z \in \mathbb{C} \quad (\mu > 0),$$

the solution of problem (1.1) can be formally presented as

$$u(t) = U(t)u_0 + \int_0^t U(t-s) f(s)\, ds \tag{2.2}$$

with

$$U(t) = E_{1+\alpha}(-t^{1+\alpha}A) = \sum_{j=0}^{\infty} \frac{1}{\Gamma\big(1+j(1+\alpha)\big)} (-t^{1+\alpha}A)^j. \tag{2.3}$$

We proceed to find an alternative expression of the solution operator $U(t)$ in (2.3). Employing the Cayley transform of the operator A

$$Q = A(I+A)^{-1}, \quad A = (I-Q)^{-1}Q,$$

we obtain

$$U(t) = E_{1+\alpha}\left(-t^{1+\alpha}(I-Q)^{-1}Q\right) = \sum_{j=0}^{\infty} \frac{1}{\Gamma\big(1+j(1+\alpha)\big)} \left(-t^{1+\alpha}(I-Q)^{-1}Q\right)^j. \tag{2.4}$$

Next, we formally substitue the operator Q in (2.4) with a scalar q and introduce the following definition.

Definition 2.2. *The functions $p_n^{(\alpha)}(t^{1+\alpha})$ produced after expanding the function $E_{1+\alpha}\Big(-\frac{q}{1-q}t^{1+\alpha}\Big)$ into a Maclaurin series in powers of q*

$$E_{1+\alpha}\Big(-\frac{q}{1-q}t^{1+\alpha}\Big) = \sum_{j=0}^{\infty}\Big(-\frac{q}{1-q}t^{1+\alpha}\Big)^j \frac{1}{\Gamma\big(1+j(1+\alpha)\big)} = \sum_{n=0}^{\infty} p_n^{(\alpha)}(t^{1+\alpha}) q^n,$$

$$p_n^{(\alpha)}(t^{1+\alpha}) = \frac{1}{n!}\frac{\partial^n}{\partial q^n} E_{1+\alpha}\Big(-\frac{q}{1-q}t^{1+\alpha}\Big)\Big|_{q=0}, \quad \alpha > -1,$$

are called the Laguerre–Cayley functions, and the polynomials $p_n^{(\alpha)}(x)$ are called the Laguerre–Cayley polynomials.

From Definition 2.2, it is easy to derive the explicit representation of the Laguerre–Cayley functions

$$p_n^{(\alpha)}(t^{1+\alpha}) = \frac{1}{n!}\frac{\partial^n}{\partial q^n}E_{1+\alpha}\Big(-\frac{q}{1-q}t^{1+\alpha}\Big)\Big|_{q=0} =$$
$$= \frac{1}{n!}\sum_{j=0}^{\infty}\frac{(-1)^j t^{j(1+\alpha)}}{\Gamma\big(1+j(1+\alpha)\big)}\frac{d^n}{dq^n}\big(q^j(1-q)^{-j}\big)\Big|_{q=0} =$$
$$= \sum_{r=0}^{n-1}\frac{(-1)^{r+1}C_{n-1}^r}{\Gamma\big(1+(r+1)(\alpha+1)\big)}t^{(r+1)(\alpha+1)}, \quad n=1,2,\dots, \qquad p_0^{(\alpha)}(t^{\alpha+1}) \equiv 1.$$

Then the Laguerre–Cayley polynomials $p_n^{(\alpha)}(x)$ can also be presented explicitly

$$p_n^{(\alpha)}(x) = \sum_{r=0}^{n-1}\frac{(-1)^{r+1}C_{n-1}^r}{\Gamma\big(1+(r+1)(\alpha+1)\big)}x^{r+1}, \quad n=1,2,\dots, \quad p_0^{(\alpha)}(x)\equiv 1,$$

where $C_n^k = \dfrac{n!}{k!(n-k)!}$ are the binomial coefficients.

Keeping Definition 2.2 in mind, we can formally express $u(t)$ in the form

$$u(t) = U(t)u_0 + \int_0^t U(s)f(t-s)\,ds, \tag{2.5}$$

with the solving operator

$$U(t) = \sum_{n=0}^{\infty}p_n^{(\alpha)}(t^{1+\alpha})Q^n, \quad Q = A(I+A)^{-1}. \tag{2.6}$$

The core idea of the Cayley transform method is to approximate the solution $u(t)$ of problem (1.1) by finite sums of the series in (2.5) and (2.6), namely:

$$u_N(t) = U_N(t)u_0 + \int_0^t U_N(s)f(t-s)\,ds, \tag{2.7}$$

where

$$\begin{gathered} U_N(t) = \sum_{n=0}^{N}p_n^{(\alpha)}(t^{1+\alpha})Q^n, \quad Q = A(I+A)^{-1}, \\ \int_0^t U_N(s)f(t-s)\,ds = \sum_{n=0}^{N}\int_0^t p_n^{(\alpha)}(s^{1+\alpha})Q^n f(t-s)\,ds. \end{gathered} \tag{2.8}$$

To proceed with the analysis, we suppose that the Laguerre–Cayley functions $p_n^{(\alpha)}(t^{1+\alpha})$ satisfy the condition

$$\big|p_n^{\alpha}(t^{1+\alpha})\big| \leqslant C(t)n^{\gamma}, \tag{2.9}$$

where $\gamma\in\mathbb{R}$ and $C(t)>0$ is independent of n. This assumption is justified by the substantial number of evaluations carried out using the computer algebra system Maple. For example, condition (2.9) is met due to the convergence of the following series:

$$\sum_{n=1}^{\infty}p_n^{(-3/4)}(7/8) = -1.1428\dots, \qquad \sum_{n=1}^{\infty}np_n^{(-1/2)}(1) = -0.5641\dots,$$
$$\sum_{n=1}^{\infty}n^2p_n^{(-3/4)}(1) = -0.3123\dots, \qquad \sum_{n=1}^{\infty}n^3p_n^{(-8/9)}(3/2) = 0.2061\dots$$

We investigate the accuracy of method (2.7), (2.8) under certain conditions on the operator A, the initial vector u_0, and the right-hand side $f(t)$. A similar study is presented in [17], where the operator A is assumed to be self-adjoint and positive definite, with eigenvectors forming an orthonormal basis in a Hilbert space.

3 The Cayley transform method without accuracy saturation

We begin by establishing two auxiliary propositions.

Lemma 3.1. *Let $k > r > 0$. Then the inequality holds*

$$\max_{t>0} \Big(\frac{t}{t+1}\Big)^k t^{-r} \leqslant \Big(\frac{r}{k}\Big)^r .$$

Proof. We consider the function $\varphi(t) = \Big(\dfrac{t}{t+1}\Big)^k t^{-r},\ \ t > 0$. Since its derivative

$$\varphi'(t) = t^{k-r-1}(t+1)^{-k-1}(k-r-rt),$$

then

$$\max_{t>0} \varphi(t) = \varphi\Big(\frac{k-r}{r}\Big) = \Big(\frac{k-r}{k}\Big)^k \Big(\frac{k-r}{r}\Big)^{-r} = \Big(\frac{k-r}{k}\Big)^{k-r} \Big(\frac{r}{k}\Big)^r < \Big(\frac{r}{k}\Big)^r.$$

The lemma is proved. □

We denote

$$v_n = Q^n u_0 = \big(A(I+A)^{-1}\big)^n u_0, \quad n = 0, 1, \dots,$$

and estimate $\|v_n\|$ in the following statement.

Lemma 3.2. *Let $\sigma > 0$, $0 < \varepsilon < \min(1; \sigma)$, $n > \sigma - \varepsilon$, $u_0 \in D(A^\sigma)$. Then the following estimate is valid*

$$\|v_n\| \leqslant \frac{C_1}{n^{(\sigma-\varepsilon)/2}} \|A^\sigma u_0\|,$$

where $C_1 = \dfrac{L}{\sin(\pi\varepsilon)} (\sigma - \varepsilon)^{(\sigma-\varepsilon)/2}$ is a positive constant.

Proof. To perform the integration, we introduce two rays Γ_+ and Γ_- on the complex plane:

$$\Gamma_\pm = \big\{ z \in \mathbb{C} \,|\, z = \rho e^{\pm i\varphi},\ \ \rho \in (0; +\infty) \big\} .$$

Then we have

$$\|v_n\| = \big\|\big(A(I+A)^{-1}\big)^n u_0\big\| = \left\| \frac{1}{2\pi i} \int_\Gamma \Big(\frac{z}{1+z}\Big)^n z^{-\sigma} (zI - A)^{-1} A^\sigma u_0 \, dz \right\| \leqslant$$
$$\leqslant \frac{L}{2\pi} \int_\Gamma \left|\frac{z}{1+z}\right|^n \frac{|z|^{-\sigma}}{1+|z|} |dz| \, \|A^\sigma u_0\|.$$

Making use of the relations

$$|z| = \big|\rho e^{\pm i\varphi}\big| = \rho, \quad |dz| = \big|e^{\pm i\varphi} d\rho\big| = d\rho,$$
$$\left|\frac{z}{1+z}\right|^2 = \left|\frac{\rho e^{\pm i\varphi}}{1+\rho e^{\pm i\varphi}}\right|^2 = \frac{\rho^2}{1 + 2\rho\cos\varphi + \rho^2} \leqslant \frac{\rho^2}{1+\rho^2},$$

we obtain

$$\|v_n\| \leqslant \frac{L}{\pi} \int_0^{+\infty} \Big(\frac{\rho^2}{1+\rho^2}\Big)^{n/2} \frac{\rho^{-\sigma}}{1+\rho} \, d\rho \, \|A^\sigma u_0\| =$$
$$= \frac{L}{\pi} \int_0^{+\infty} \Big(\frac{\rho^2}{1+\rho^2}\Big)^{n/2} (\rho^2)^{-(\sigma-\varepsilon)/2} \frac{\rho^{-\varepsilon} d\rho}{1+\rho} \, \|A^\sigma u_0\| \leqslant$$

$$\leqslant \frac{L}{\pi}\sup_{t>0}\left\{\left(\frac{t}{1+t}\right)^{n/2} t^{-(\sigma-\varepsilon)/2}\right\}\int_0^{+\infty}\frac{\rho^{-\varepsilon}d\rho}{1+\rho}\,\|A^\sigma u_0\|.$$

Applying Lemma 3.2 with $k=\dfrac{n}{2}$ and $r=\dfrac{\sigma-\varepsilon}{2}$ and evaluating the improper integral

$$\int_0^{+\infty}\frac{\rho^{-\varepsilon}d\rho}{1+\rho}=B(1-\varepsilon,\varepsilon)=\frac{\pi}{\sin\pi\varepsilon},$$

where $B(p,q)$ is the Beta function, we deduce the inequality

$$\|v_n\|\leqslant\frac{L}{\pi}\left(\frac{\sigma-\varepsilon}{n}\right)^{(\sigma-\varepsilon)/2}\frac{\pi}{\sin(\pi\varepsilon)},$$

thus completing the proof of the lemma. □

Now we turn to investigating the accuracy of the approximate solution $u_N(t)$ in the case of the homogeneous equation (1.1). Then formulas (2.5) and (2.7) become as follows:

$$u(t)=U(t)u_0=\sum_{n=0}^{\infty}p_n^{(\alpha)}(t^{1+\alpha})v_n, \tag{3.1}$$

$$u_N(t)=U_N(t)u_0=\sum_{n=0}^{N}p_n^{(\alpha)}(t^{1+\alpha})v_n, \quad v_n=Q^n u_0. \tag{3.2}$$

Theorem 3.3. *Let $f(t)\equiv 0$, the Laguerre–Cayley functions $p_n^{(\alpha)}(t^{1+\alpha})$ satisfy condition (2.9), $u_0\in D(A^\sigma)$ with $\sigma>\max\{0;2(\gamma+1)\}$. Then, for all ε such that $0<\varepsilon<\min\{1;\sigma-2(\gamma+1)\}$, and all integers N such that $N+1\geqslant\sigma$, the following estimate holds*

$$\|u(t)-u_N(t)\|\leqslant\frac{M_1(t)}{N^{\frac{\sigma-\varepsilon}{2}-\gamma-1}}\|A^\sigma u_0\|, \tag{3.3}$$

where $M_1(t)=\dfrac{2C(t)C_1}{\sigma-\varepsilon-2(\gamma+1)}>0$ is independent of N.

Proof. Using representations (3.1) and (3.2), and applying Lemmas 3.1 and 3.2, yields the chain of inequalities:

$$\|u(t)-u_N(t)\|=\left\|\sum_{n=N+1}^{\infty}p_n^{(\alpha)}(t^{1+\alpha})v_n\right\|\leqslant\sum_{n=N+1}^{\infty}\left|p_n^{(\alpha)}(t^{1+\alpha})\right|\|v_n\|\leqslant$$

$$\leqslant\sum_{n=N+1}^{\infty}C(t)n^\gamma\frac{C_1}{n^{\frac{\sigma-\varepsilon}{2}}}\|A^\sigma u_0\|\leqslant C(t)C_1\int_N^\infty\frac{dx}{x^{\frac{\sigma-\varepsilon}{2}-\gamma}}\|A^\sigma u_0\|=$$

$$=\frac{C(t)C_1}{\frac{\sigma-\varepsilon}{2}-\gamma-1}\frac{1}{N^{\frac{\sigma-\varepsilon}{2}-\gamma-1}}\|A^\sigma u_0\|,$$

which immediately leads to (3.3) and hence establishes the lemma. □

Next, we study the accuracy of the approximate solution $u_N(t)$ for the inhomogeneous equation (1.1) subject to the zero initial vector $u_0=0$. Then formulas (2.5) and (2.7) reduce to

$$u(t)=\int_0^t\sum_{n=0}^{\infty}p_n^{(\alpha)}(s^{1+\alpha})Q^n f(t-s)\,ds, \tag{3.4}$$

$$u_N(t)=\int_0^t\sum_{n=0}^{N}p_n^{(\alpha)}(s^{1+\alpha})Q^n f(t-s)\,ds. \tag{3.5}$$

Theorem 3.4. *Let $u_0 = 0$, the Laguerre–Cayley functions $p_n^{(\alpha)}(t^{1+\alpha})$ satisfy inequality (2.9) with $\int_0^t C^2(s)\,ds < \infty$, and the right-hand side $f(t)$ meet the conditions*

$$f(t) \in D(A^\sigma),\ \sigma > \max\big\{0; 2(\gamma+1)\big\}; \qquad \int_0^t \|A^\sigma f(s)\|^2\,ds < \infty.$$

Then, for all ε such that $0 < \varepsilon < \min\big\{1; \sigma - 2(\gamma+1)\big\}$, and all integers N such that $N + 1 \geqslant \sigma$, the following estimate holds

$$\|u(t) - u_N(t)\| \leqslant \frac{M_2(t)}{N^{\frac{\sigma-\varepsilon}{2}-\gamma-1}} \left[\int_0^t \|A^\sigma f(s)\|^2 ds\right]^{1/2}, \tag{3.6}$$

where $M_2(t) = \dfrac{2C_1}{\sigma - \varepsilon - 2(\gamma+1)}\left[\displaystyle\int_0^t C^2(s)\,ds\right]^{1/2} \geqslant 0$ is independent of N.

Proof. Taking into account representations (3.4) and (3.5), utilizing the result of Lemmas 3.1 and the technique from Lemma 3.2, and applying the Cauchy–Bunyakovsky inequality, we obtain

$$\|u(t) - u_N(t)\| \leqslant \left\| \int_0^t \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(s^{1+\alpha}) Q^n f(t-s)\,ds \right\| \leqslant$$

$$\leqslant \sum_{n=N+1}^{\infty} \int_0^t \big|p_n^{(\alpha)}(s^{1+\alpha})\big| \|Q^n f(t-s)\|\,ds \leqslant$$

$$\leqslant \sum_{n=N+1}^{\infty} \frac{C_1}{n^{\frac{\sigma-\varepsilon}{2}-\gamma}} \int_0^t C(s)\,\|A^\sigma f(t-s)\|\,ds \leqslant$$

$$\leqslant C_1 \int_N^{+\infty} \frac{dx}{x^{\frac{\sigma-\varepsilon}{2}-\gamma}} \left[\int_0^t C^2(s) ds\right]^{1/2} \left[\int_0^t \|A^\sigma(s)\|^2 ds\right]^{1/2}.$$

The evaluation of the integral

$$\int_N^{+\infty} \frac{dx}{x^{\frac{\sigma-\varepsilon}{2}-\gamma}} = \frac{1}{\frac{\sigma-\varepsilon}{2} - \gamma - 1} \frac{1}{N^{\frac{\sigma-\varepsilon}{2}-\gamma-1}}$$

leads to estimate (3.6) and completes the proof of the theorem. □

Finally, Theorems 3.3 and 3.4 together yield the following main result.

Theorem 3.5. *Let the assumptions of Theorems 3.3 and 3.4 hold. Then, the Cayley transform method given by (2.7), (2.8) is free from accuracy saturation and is characterized by the following error estimate*

$$\|u(t) - u_N(t)\| \leqslant \frac{M(t)}{N^{\frac{\sigma-\varepsilon}{2}-\gamma-1}} \left\{ \|A^\sigma u_0\| + \left[\int_0^t \|A^\sigma f(s)\|^2\,ds\right]^{1/2} \right\},$$

where $M(t) = \max\big\{M_1(t); M_2(t)\big\} = \dfrac{2C_1}{\sigma - \varepsilon - 2(\gamma+1)} \max\left\{ C(t); \left[\int_0^t C^2(s)\,ds\right]^{1/2} \right\} > 0$ is independent of N.

4 Numerical results

To validate our approach in practice, we present several examples below for the case of $\alpha = -1/2$.

Example 4.1. We consider the scalar problem (1.1) with $A = 1$, $f(t) \equiv 0$, $u_0 = 1$

$$u'(t) + \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{u(s)\, ds}{\sqrt{t-s}} = 0, \quad t > 0, \quad u(0) = 1.$$

To evaluate the exact solution $u(t)$ at $t = 1/3$ and $t = 5$, we use formula (2.2)

$$u(1/3) = E_{1/2}(-\sqrt{1/3}\,) = \sum_{j=0}^{\infty} \frac{(-\sqrt{1/3}\,)^j}{\Gamma(1 + j/2)} = e^{1/3}\big(1 - \operatorname{erf}(\sqrt{1/3}\,)\big) = 0{,}57808524502694239\ldots,$$

$$u(5) = E_{1/2}(-\sqrt{5}\,) = \sum_{j=0}^{\infty} \frac{(-\sqrt{5}\,)^j}{\Gamma(1 + j/2)} = e^{5}\big(1 - \operatorname{erf}(\sqrt{5}\,)\big) = 0{,}23232629437646507\ldots,$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-t^2} dt$ is the error function.

To evaluate the approximate solution $u_N(t)$ at $t = 1/3$ and $t = 5$, we apply formula (3.2) with $Q = A/(1 + A)^{-1} = 1/2$

$$u_N(1/3) = \sum_{n=0}^{N} \frac{1}{2^n} p_n^{(-1/2)}(\sqrt{1/3}\,) = 1 + \sum_{n=1}^{N} \frac{1}{2^n} \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^r (1/3)^{(r+1)/2}}{\Gamma\big(1 + (r+1)/2\big)},$$

$$u_N(5) = \sum_{n=0}^{N} \frac{1}{2^n} p_n^{(-1/2)}(\sqrt{5}\,) = 1 + \sum_{n=1}^{N} \frac{1}{2^n} \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^r 5^{(r+1)/2}}{\Gamma\big(1 + (r+1)/2\big)}.$$

The values of the absolute errors $err_N(1/3)$ and $err_N(5)$ for various N are presented in Tables 4.1 and 4.2, respectively.

Table 4.1. Example 4.1 for $t = 1/3$

N	$u_N(1/3)$	$\mathrm{err}_N(1/3)$
8	0,57803266839633512 . . .	$5{,}257\ldots \cdot 10^{-5}$
16	0,57808528523816720 . . .	$4{,}021\ldots \cdot 10^{-8}$
32	0,57808524502693419 . . .	$8{,}192\ldots \cdot 10^{-15}$
64	0,57808524502694239 . . .	$6{,}645\ldots \cdot 10^{-26}$

Table 4.2. Example 4.1 for $t = 5$

N	$u_N(5)$	$\mathrm{err}_N(5)$
8	0,23233906355547200 . . .	$1{,}276\ldots \cdot 10^{-5}$
16	0,23232628934289949 . . .	$5{,}033\ldots \cdot 10^{-9}$
32	0,23232629437646504 . . .	$2{,}652\ldots \cdot 10^{-17}$
64	0,23232629437646507 . . .	$5{,}845\ldots \cdot 10^{-30}$

Example 4.2. We study the scalar problem (1.1) with $A = 1$, $f(t) = \sqrt{t}$, $u_0 = 0$

$$u'(t) + \frac{d}{dt} \frac{1}{\sqrt{\pi}} \int_0^t \frac{u(s)\, ds}{\sqrt{t-s}} = \sqrt{t}, \quad t > 0, \quad u(0) = 0.$$

To evaluate the exact solution $u(t)$ at $t = 1/3$ and $t = 5$, we use formula (2.2)

$$u(1/3) = \int_0^{1/3} E_{1/2}(-\sqrt{s}) f(1/3 - s)\, ds = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma\big(1 + j(1+\alpha)\big)} \int_0^{1/3} s^{j/2} \sqrt{1/3 - s}\, ds =$$

$$= \int_0^{1/3} e^s \big(1 - \mathrm{erf}(\sqrt{s})\big) \sqrt{1/3 - s}\, ds = 0{,}09197092206422016\ldots,$$

$$u(5) = \int_0^{5} E_{1/2}(-\sqrt{s}) f(5 - s)\, ds = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma\big(1 + j(1+\alpha)\big)} \int_0^{5} s^{j/2} \sqrt{5 - s}\, ds =$$

$$= \int_0^{5} e^s \big(1 - \mathrm{erf}(\sqrt{s})\big) \sqrt{5 - s}\, ds = 2{,}87539975764967135\ldots$$

To evaluate the approximate solution $u_N(t)$ at $t = 1/3$ and $t = 5$, we apply formula (3.5) with $Q = A/(1+A)^{-1} = 1/2$

$$u_N(1/3) = \int_0^{1/3} U_N(s) f(1/3 - s) ds = \sum_{n=0}^{N} \frac{1}{2^n} \int_0^{1/3} p_n^{(-1/2)}(\sqrt{s}) \sqrt{1/3 - s} ds =$$

$$= \int_0^{1/3} \sqrt{1/3 - s}\, ds + \sum_{n=1}^{N} \frac{1}{2^n} \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^r}{\Gamma\big(1 + (r+1)/2\big)} \int_0^{1/3} s^{(r+1)/2} \sqrt{1/3 - s}\, ds,$$

$$u_N(5) = \int_0^{5} U_N(s) f(5 - s) ds = \sum_{n=0}^{N} \frac{1}{2^n} \int_0^{5} p_n^{(-1/2)}(\sqrt{s}) \sqrt{5 - s} ds =$$

$$= \int_0^{5} \sqrt{5 - s}\, ds + \sum_{n=1}^{N} \frac{1}{2^n} \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^r}{\Gamma\big(1 + (r+1)/2\big)} \int_0^{5} s^{(r+1)/2} \sqrt{5 - s}\, ds.$$

The values of the absolute errors $err_N(1/3)$ and $err_N(5)$ for various N are presented in Tables 4.3 and 4.4, respectively.

Table 4.3. Example 4.2 for $t = 1/3$

N	$u_N(1/3)$	$\mathrm{err}_N(1/3)$
8	$0{,}09197108639915791\ldots$	$1{,}643\ldots \cdot 10^{-7}$
16	$0{,}09197092122394625\ldots$	$8{,}402\ldots \cdot 10^{-10}$
32	$0{,}09197092206421994\ldots$	$2{,}194\ldots \cdot 10^{-16}$
64	$0{,}09197092206422016\ldots$	$9{,}842\ldots \cdot 10^{-28}$

Table 4.4. Example 4.2 for $t = 5$

N	$u_N(5)$	$\mathrm{err}_N(5)$
8	$2{,}87542518637049133\ldots$	$2{,}542\ldots \cdot 10^{-5}$
16	$2{,}87539975959779992\ldots$	$1{,}948\ldots \cdot 10^{-9}$
32	$2{,}87539975764967140\ldots$	$4{,}830\ldots \cdot 10^{-17}$
64	$2{,}87539975764967135\ldots$	$6{,}256\ldots \cdot 10^{-32}$

5 Conclusion

Theorem 3.5 substantiates that the Cayley transform method, defined by formulas (2.7) and (2.8), has a power-type convergence rate close to $O\left(\frac{1}{N^{\sigma/2-\gamma-1}}\right)$. This estimate indicates that the method does not exhibit accuracy saturation: its accuracy naturally adapts to the regularity of the right-hand side $f(t)$ and the initial vector u_0. The parameter σ reflects the smoothness of this data, while γ is related to the properties of the Laguerre–Cayley polynomials given by inequality (2.9), the study of which remains an area of ongoing research.

The numerical experiments include both exact and approximate solutions of the homogeneous and inhomogeneous equations at two time points, $t = 1/3$ and $t = 5$. The results are satisfactory even for small values of the parameter N – the number of terms in the partial sum of the series involved. The case $\alpha = -1/2$ is considered purely for simplicity and without loss of generality.

Declarations

Conflict of Interest.
There is no conflict of interest.

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