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Error analysis of truncation Legendre method for solving numerical differentiation

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Анотація. Ми розглядаємо задачу чисельного диференціювання функцій із зважених класів Вінера, яка є фундаментальною в обчислювальній математиці та спричиняє суттєві труднощі через її некоректність: малі збурення у вхідних даних можуть призводити до значних похибок в наближенні похідних. Ми будуємо та аналізуємо метод усічених рядів Лежандра для знаходження похідних довільного порядку, виходячи за межі традиційного фокусу лише на похідних першого порядку. Метод ґрунтується на ортогональних властивостях поліномів Лежандра та використовує параметр усічення, який забезпечує баланс між похибкою апроксимації та збуреннями у вхідних даних. Наш основний внесок полягає в отриманні явних оцінок похибки як у інтегральних, так і в рівномірних метриках, що суттєво узагальнює попередні результати, обмежені певними випадками. Ми встановлюємо оптимальний вибір параметра усічення як функції від рівня шуму, параметра гладкості та порядку похідної. Аналіз охоплює широкий спектр параметрів, що характеризують клас функцій, порядок похідної та метрики як вхідного, так і вихідного просторів. Ми довели точність запропонованого методу в зваженому класі Вінера, з наданням явних порядків збіжності. Запропонований нами теоретичний підхід дозволяє практикам визначати оптимальний баланс між обчислювальною складністю та точністю для конкретних застосувань.

Abstract. We study the problem of numerical differentiation of functions from weighted Wiener classes. This problem is fundamental in computational mathematics and presents significant challenges due to its ill-posed nature, where small perturbations in input data can lead to substantial errors in derivative approximations. We construct and analyze a truncation Legendre method to recover arbitrary order derivatives, extending beyond the traditional focus on first-order derivatives. The method utilizes the orthogonal properties of Legendre polynomials and employs a truncation parameter that balances the approximation error with the propagation of input data perturbations. Our main contribution is obtaining explicit error estimates in both integral and uniform metrics, significantly generalizing previous results that were limited to specific cases. We establish optimal choices for the truncation parameter as a function of the noise level, smoothness parameter, and the order of derivative. The analysis covers a comprehensive range of parameters characterizing the function class, derivative order, and metrics of both input and output spaces. We prove the accuracy of the proposed method in the weighted Wiener class, providing explicit convergence rates. Our theoretical framework enables practitioners to determine the optimal balance between computational complexity and accuracy for their specific applications.

Key words: Numerical differentiation, Legendre polynomials, truncation method, weighted Wiener classes, error estimates.

1 Introduction

The problem of numerical differentiation poses a central challenge in computational mathematics, with widespread applications across scientific research and engineering disciplines. This challenge gains particular prominence in situations where analytical computation of function derivatives presents difficulties or proves impossible, especially when working with approximate or noisy functional data. It is noteworthy that numerical differentiation, despite its extensive historical development, continues to attract significant attention within the mathematical research community (see, for example, [1] [2], [4], [6], [7], [10–13], [19], [21]).

The fundamental challenge associated with numerical differentiation lies in its inherent instability, rendering it ill-posed according to Hadamard's definition. Specifically, small perturbations in input data can result in substantial errors when evaluating the deviation of approximate derivatives from exact ones.

The present investigation focuses on the truncated Legendre method for addressing numerical differentiation problems. It is important to note that this method was initially examined in the context of numerical differentiation in [20] and [9]. This approach has demonstrated several beneficial characteristics, notably its straightforward algorithmic implementation and the elimination of requirements for solving equation systems. Consequently, research of the truncated Legendre method has been extended to various classes of differentiable functions (see, e.g., [14], [15], [17]).

The aim of our investigation is to derive error bounds for this method when applied to functions belonging to weighted Wiener classes for the reconstruction of their derivatives of arbitrary order. In contrast to earlier investigations, which predominantly concentrated on first-order derivatives and their reconstruction in the metric of specific C and L_2 spaces, we conduct a thorough analysis across an extensive range of parameters characterizing the function class under investigation, derivative order, and metrics of both input and output spaces.

In developing our proposed approach, particular emphasis is placed on identifying the optimal value of the truncation parameter, which minimizes the combined error arising from series truncation and input data perturbations. We establish explicit relationships between this parameter and the input data error level, function smoothness, and the order of the derivative.

Our findings possess both theoretical importance, which is crucial for comprehending the fundamental nature of numerical differentiation problems, and practical significance in creating stable computational algorithms for diverse applied problems.

We note that our investigation generalizes existing results to encompass derivatives of arbitrary order and various output metrics (integral and uniform), enabling a more comprehensive characterization of the capabilities and effective application conditions of the truncated expansion method for numerical differentiation.

To establish our theoretical framework, we introduce the essential mathematical concepts. We denote by L_q, $2 \le q \le \infty$, the space of real-valued functions $f(t)$ that are integrable to the q-th power on the interval $[-1,1]$, with norms specified as

$$\|f\|_q^q := \int_{-1}^{1} |f(t)|^q \, dt < \infty, \quad 2 \le q < \infty,$$

$$\|f\|_\infty := \operatorname*{ess\,sup}_{t\in[-1,1]} |f(t)| < \infty, \quad q = \infty.$$

We employ the system of orthonormal Legendre polynomials $\{\varphi_k(t)\}_{k=0}^{\infty}$ given by

$$\varphi_k(t) = \sqrt{k+1/2}(2^k k!)^{-1} \frac{d^k}{dt^k}[(t^2-1)^k], \quad k = 0,1,2,\dots.$$

In the Hilbert space L_2, the norm may be represented using these Legendre polynomials as

$$\|f\|_2^2 = \sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2,$$

where the Fourier-Legendre coefficients are defined by

$$\langle f, \varphi_k \rangle = \int_{-1}^{1} f(t)\, \varphi_k(t)\, dt, \quad k = 0, 1, 2, \ldots.$$

Furthermore, we denote C as the space of continuous real-valued functions on $[-1, 1]$ and ℓ_p, $1 \le p \le \infty$, as the space of numerical sequences $\overline{x} = \{x_k\}_{k \in \mathbb{N}_0}$ that satisfy

$$\|\overline{x}\|_{\ell_p} := \begin{cases} \Big(\sum\limits_{k \in \mathbb{N}_0} |x_k|^p \Big)^{\frac{1}{p}} < \infty, & 1 \le p < \infty, \\ \sup\limits_{k \in \mathbb{N}_0} |x_k| < \infty, & p = \infty. \end{cases}$$

For our theoretical analysis, we introduce the functional spaces

$$W_s^\mu = \{ f \in L_2 : \quad \|f\|_{s,\mu}^s := \sum_{k=0}^{\infty} (\max\{1, k\})^{s\mu} |\langle f, \varphi_k \rangle|^s < \infty \},$$

where $\mu > 0$ and $1 \le s < \infty$. We employ the same notation for both the space and its unit ball: $W_s^\mu = \{ f \in W_s^\mu : \|f\|_{s,\mu} \le 1 \}$, which we term a function class. The specific interpretation, space or class, will be clear from the context. These functional spaces W_s^μ are extensively studied in approximation theory and are recognized as weighted Wiener classes (as discussed in [8]).

Remark 1.1. *We introduce L_2^r, $r = 1, 2, \ldots$, as the space of functions $f(t)$, where $f, f', \ldots, f^{(r-1)}$ demonstrate absolute continuity on any interval (η_1, η_2) with $\eta_1 > -1$ and $\eta_2 < 1$, and satisfy*

$$\int_{-1}^{1} (1-t)^r (1+t)^r |f^{(r)}(t)|^2 \, dt < \infty.$$

Based on [3], the space L_2^r can be endowed with the norm

$$\|f\|_{L_2^r} := \Big(\sum_{j=0}^{r} \int_{-1}^{1} (1-t)^j (1+t)^j |f^{(j)}(t)|^2 dt \Big)^{1/2}$$

and the following equivalence is established 1) $f \in L_2^r$ if and only if 2) $f \in W_2^r$.

Any function $f(t)$ from W_s^μ admits the representation

$$f(t) = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k(t),$$

while its r-th order derivative with $r \in \mathbb{N}$ can be written as

$$f^{(r)}(t) = \sum_{k=r}^{\infty} \langle f, \varphi_k \rangle \varphi_k^{(r)}(t). \tag{1.1}$$

In our problem setup, we assume that the input error measurements are conducted using the ℓ_p metric. More precisely, we will deal with a sequence of real numbers $\overline{f^\delta} = \{\langle f^\delta, \varphi_k \rangle\}_{k \in \mathbb{N}_0}$ such that, for $\overline{\xi} = \{\xi_k\}_{k \in \mathbb{N}_0}$, $\xi_k = \langle f - f^\delta, \varphi_k \rangle$, and for some $1 \le p \le \infty$, the following constraint is satisfied

$$\|\overline{\xi}\|_{\ell_p} \le \delta, \quad 0 < \delta < 1. \tag{1.2}$$

The paper is organized as follows. In Sections 2 and 3, we analyze the truncation method and derive its error estimates in the uniform and L_2 metrics, respectively. In Section 4, we provide a comprehensive analysis for the general L_q-metrics with $2 \le q \le \infty$, unifying and extending the results from the previous sections.

2 Truncation method. Error estimate in the uniform metrics

For reconstructing the r-th derivative (1.1) of functions $f \in W_s^\mu$, $r = 1, 2, \ldots$, we implement the truncation method

$$\mathcal{D}_N^{(r)} f^\delta(t) = \sum_{k=r}^{N} \langle f^\delta, \varphi_k \rangle \varphi_k^{(r)}(t). \tag{2.1}$$

For our calculations, we need the following formula for differentiation of Legendre polynomials

$$\varphi_k'(t) = 2\sqrt{k+1/2} \sum_{l=0}^{k-1}{}^{*} \sqrt{l+1/2}\,\varphi_l(t), \quad k \in \mathbb{N}, \tag{2.2}$$

where in aggregate $\sum\limits_{l=0}^{k-1}{}^{*} \bar{\varphi}_l(t)$ the summation is extended over only those terms for which $k+l$ is odd.

Let us write the error of the method (2.1) as

$$f^{(r)}(t) - \mathcal{D}_N^{(r)} f^\delta(t) = \left(f^{(r)}(t) - \mathcal{D}_N^{(r)} f(t)\right) + \left(\mathcal{D}_N^{(r)} f(t) - \mathcal{D}_N^{(r)} f^\delta(t)\right) \tag{2.3}$$

with

$$\mathcal{D}_N^{(r)} f(t) = \sum_{k=r}^{N} \langle f, \varphi_k \rangle \varphi_k^{(r)}(t).$$

The parameter N should be chosen depending on δ, p, s and μ so as to minimize the error estimate for the method (2.1).

We use the notation $A \asymp B$ when there are absolute constants $C_1, C_2 > 0$ such that $C_1 B \le A \le C_2 B$ is true. We adopt the convention that c denotes a generic positive constant, which can vary from inequality to inequality and does not depend on the values of δ, N, or the function f.

An upper bound for the norm of the first difference on the right-hand side of (2.3) is contained in the following statement.

Lemma 2.1. *Let $f \in W_s^\mu$, $1 \le s < \infty$, $\mu > 2r - 1/s + 3/2$. Then it holds*

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f\|_C \le c\|f\|_{s,\mu} N^{-\mu+2r-1/s+3/2}.$$

Proof. Using (2.2), we write down the representation

$$f^{(r)}(t) - \mathcal{D}_N^{(r)} f(t) = \sum_{k=N+1}^{\infty} \langle f, \varphi_k \rangle \varphi_k^{(r)}(t) = 2 \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\,\langle f, \varphi_k \rangle \sum_{l_1=0}^{k-1}{}^{*} \sqrt{l_1+1/2}\,\varphi_{l_1}^{(r-1)}(t) = \ldots$$

$$= 2^r \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\,\langle f, \varphi_k \rangle \sum_{l_1=r-1}^{k-1}{}^{*} (l_1+1/2) \sum_{l_2=r-2}^{l_1-1}{}^{*} (l_2+1/2)$$

$$\ldots \sum_{l_{r-1}=1}^{l_{r-2}-1}{}^{*} (l_{r-1}+1/2) \sum_{l_r=0}^{l_{r-1}-1}{}^{*} \sqrt{l_r+1/2}\,\varphi_{l_r}(t).$$

It is well-known (see for example [18], Theorem 7.3.1) that

$$\max_{-1\le t\le 1} |\varphi_k(t)| = \varphi_k(1) := \sqrt{k+1/2}, \quad k = 0, 1, 2, \ \ldots.$$

Now we can estimate

$$\|f^{(r)} - D_N^{(r)} f\|_C \le 2^r \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\,|\langle f,\varphi_k\rangle| \sum_{l_1=r-1}^{k-1} (l_1+1/2) \sum_{l_2=r-2}^{l_1-1} (l_2+1/2)$$

$$\dots \sum_{l_{r-2}=1}^{l_{r-3}-1} (l_{r-2}+1/2) \sum_{l_{r-1}=1}^{l_{r-2}-1} (l_{r-1}+1/2) \sum_{l_r=0}^{l_{r-1}-1} \sqrt{l_r+1/2}\,\|\varphi_{l_r}\|_C$$

$$\le 2^r \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\,|\langle f,\varphi_k\rangle| \sum_{l_1=r-1}^{k-1} (l_1+1/2) \sum_{l_2=r-2}^{l_1-1} (l_2+1/2)$$

$$\dots \sum_{l_{r-2}=1}^{l_{r-3}-1} (l_{r-2}+1/2) \sum_{l_{r-1}=1}^{l_{r-2}-1} (l_{r-1}+1/2) \sum_{l_r=0}^{l_{r-1}-1} (l_r+1/2)$$

$$\le c \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\; k^{2r}|\langle f,\varphi_k\rangle| = c \sum_{k=N+1}^{\infty} k^{2r+1/2}|\langle f,\varphi_k\rangle|.$$

Let's start with the case $1 < s < \infty$. Then using the Hölder inequality and the definition of W_s^μ, we continue for $\mu > 2r - 1/s + 3/2$

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f\|_C \le c \sum_{k=N+1}^{\infty} k^{\mu}|\langle f,\varphi_k\rangle| k^{2r+1/2-\mu}$$

$$\le c \left(\sum_{k=N+1}^{\infty} k^{s\mu}|\langle f,\varphi_k\rangle|^s \right)^{1/s} \left(\sum_{k=N+1}^{\infty} k^{(2r+1/2-\mu)\frac{s}{s-1}} \right)^{(s-1)/s}$$

$$\le c\|f\|_{s,\mu} N^{-\mu+2r-1/s+3/2}.$$

In the case of $s=1$ the assertion of Lemma is proved similarly. □

Lemma 2.2. *Let the condition (1.2) be satisfied for $1 \le p \le \infty$. Then for arbitrary function $f \in L_2$ it holds*

$$\|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_C \le c\delta N^{2r-1/p+3/2}.$$

Proof. Using (2.2), we write down the representation

$$\mathcal{D}_N^{(r)} f(t) - \mathcal{D}_N^{(r)} f^\delta(t) = \sum_{k=r}^{N} \langle f - f^\delta, \varphi_k\rangle \varphi_k^{(r)}(t) = 2\sum_{k=r}^{N} \sqrt{k+1/2}\,\langle f - f^\delta, \varphi_k\rangle \sum_{l_1=0}^{k-1}{}^{*}\, \varphi_{l_1}^{(r-1)}(t)$$

$$\dots = 2^r \sum_{k=r}^{N} \sqrt{k+1/2}\,\langle f - f^\delta, \varphi_k\rangle \sum_{l_1=r-1}^{k-1}{}^{*} (l_1+1/2) \sum_{l_2=r-2}^{l_1-1}{}^{*} (l_2+1/2)$$

$$\dots \sum_{l_{r-1}=1}^{l_{r-2}-1}{}^{*} (l_{r-1}+1/2) \sum_{l_r=0}^{l_{r-1}-1}{}^{*} \sqrt{l_r+1/2}\; \varphi_{l_r}(t). \tag{2.4}$$

Let $1 < p < \infty$ first. Then, using the Hölder inequality, we find

$$\|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_C \le 2^r \sum_{k=r}^{N} \sqrt{k+1/2}\,|\langle f - f^\delta, \varphi_k\rangle| \sum_{l_1=r-1}^{k-1} (l_1+1/2) \sum_{l_2=r-2}^{l_1-1} (l_2+1/2)$$

$$\dots \sum_{l_{r-1}=1}^{l_{r-2}-1} (l_{r-1}+1/2) \sum_{l_r=0}^{l_{r-1}-1} (l_r+1/2)$$

$$\leq c \sum_{k=r}^{N} k^{2r+1/2} \left| \langle f - f^\delta, \varphi_k \rangle \right| \leq c \left(\sum_{k=r}^{N} |\xi_k|^p \right)^{1/p} \left(\sum_{k=r}^{N} k^{(2r+1/2)\frac{p}{p-1}} \right)^{(p-1)/p}$$

$$\leq c\delta N^{2r-1/p+3/2},$$

which was required to prove.

In the cases of $p = 1$ and $p = \infty$, the assertion of Lemma is proved similarly. □

The combination of Lemmas 2.1 and 2.2 gives

Theorem 2.3. *Let $f \in W_s^\mu$, $1 \leq s < \infty$, $\mu > 2r - 1/s + 3/2$, and let the condition (1.2) be satisfied for $1 \leq p \leq \infty$. Then for $N \asymp \delta^{-\frac{1}{\mu - 1/p + 1/s}}$ it holds*

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f^\delta\|_C \leq c\delta^{\frac{\mu - 2r + 1/s - 3/2}{\mu - 1/p + 1/s}}.$$

Proof. Taking into account Lemmas 2.1, 2.2, from (2.3) we get

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f^\delta\|_C \leq \|f^{(r)} - \mathcal{D}_N^{(r)} f\|_C + \|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_C$$

$$\leq c\|f\|_{s,\mu} N^{-\mu+2r-1/s+3/2} + c\delta N^{2r-1/p+3/2} = cN^{2r+3/2} \left(N^{-\mu-1/s} + \delta N^{-1/p} \right).$$

Substituting the rule $N \asymp \delta^{-\frac{1}{\mu - 1/p + 1/s}}$ into the relation above completely proves Theorem. □

In what follows, by $\mathrm{card}([r, N])$ we shall understand the number of points with integer indexes that belong to the interval $[r, N]$.

Corollary 2.4. *In the considered problem, the truncation method $\mathcal{D}_N^{(r)}$ (2.1) achieves the accuracy*

$$O\left(\delta^{\frac{\mu - 2r + 1/s - 3/2}{\mu - 1/p + 1/s}} \right)$$

on the class W_s^μ, $\mu > 2r - 1/s + 3/2$, and requires

$$\mathrm{card}([r, N]) \asymp N \asymp \delta^{-\frac{1}{\mu - 1/p + 1/s}}$$

perturbed Fourier-Legendre coefficients.

3 Error estimate for truncation method in L_2–metric

Now we have to bound the error of the method (2.1) in the metric of L_2. An upper estimate for the norm of the first difference on the right-hand side of (2.3) is contained in the following statement.

Lemma 3.1. *Let $f \in W_s^\mu$, $1 \leq s < \infty$, $\mu > 2r - 1/s + 1/2$. Then it holds*

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f\|_2 \leq c\|f\|_{s,\mu} N^{-\mu+2r-1/s+1/2}.$$

Proof. We take the representation

$$f^{(r)}(t) - \mathcal{D}_N^{(r)} f(t)$$

$$= 2^r \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\, \langle f, \varphi_k \rangle \sideset{}{^*}\sum_{l_1=r-1}^{k-1} (l_1 + 1/2) \sideset{}{^*}\sum_{l_2=r-2}^{l_1-1} (l_2 + 1/2) \cdots \sideset{}{^*}\sum_{l_r=0}^{l_{r-1}-1} \sqrt{l_r + 1/2}\, \varphi_{l_r}(t)$$

and change the order of summation

$$f^{(r)}(t) - \mathcal{D}_N^{(r)} f(t) = 2^r \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\langle f, \varphi_k\rangle \sideset{}{^*}\sum_{l_1=r-1}^{k-1} (l_1+1/2)$$

$$\ldots \sideset{}{^*}\sum_{l_{r-1}=1}^{l_{r-2}-1} (l_{r-1}+1/2) \sideset{}{^*}\sum_{l_r=0}^{l_{r-1}-1} \sqrt{l_r+1/2}\, \varphi_{l_r}(t) = 2^r \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\, \langle f, \varphi_k\rangle$$

$$\ldots \sideset{}{^*}\sum_{l_{r-2}=2}^{l_{r-3}-1} (l_{r-2}+1/2) \sideset{}{^*}\sum_{l_r=0}^{l_{r-2}-1} \sqrt{l_r+1/2}\, \varphi_{l_r}(t) \sideset{}{^*}\sum_{l_{r-1}=l_r+1}^{l_{r-2}-1} (l_{r-1}+1/2)$$

$$= \ldots = 2^r \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\, \langle f,\, \varphi_k\rangle \sideset{}{^*}\sum_{l_1=r-1}^{k-1} (l_1+1/2) \sideset{}{^*}\sum_{l_r=0}^{l_1-r+1} \sqrt{l_r+1/2}\,\varphi_{l_r}(t) \sideset{}{^*}\sum_{l_2=l_r+r-2}^{l_1-1} (l_2+1/2)$$

$$\ldots \sideset{}{^*}\sum_{l_{r-2}=l_r+2}^{l_{r-3}-1} (l_{r-2}+1/2) \sideset{}{^*}\sum_{l_{r-1}=l_r+1}^{l_{r-2}-1} (l_{r-1}+1/2)$$

$$= 2^r \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\, \langle f, \varphi_k\rangle \sideset{}{^*}\sum_{l_r=0}^{k-r} \sqrt{l_r+1/2}\, \varphi_{l_r}(t) \sideset{}{^*}\sum_{l_1=l_r+r-1}^{k-1} (l_1+1/2) \sideset{}{^*}\sum_{l_2=l_r+r-2}^{l_1-1} (l_2+1/2)$$

$$\ldots \sideset{}{^*}\sum_{l_{r-2}=l_r+2}^{l_{r-3}-1} (l_{r-2}+1/2) \sideset{}{^*}\sum_{l_{r-1}=l_r+1}^{l_{r-2}-1} (l_{r-1}+1/2).$$

By interchanging the order of summation with respect to k and l_r, we obtain

$$f^{(r)}(t) - \mathcal{D}_N^{(r)} f(t) = 2^r \sideset{}{^*}\sum_{l_r=0}^{N-r+1} \sqrt{l_r+1/2}\, \varphi_{l_r}(t) \sum_{k=N+1}^{\infty} \sqrt{k+1/2}\langle f, \varphi_k\rangle B_k^r$$

$$+2^r \sideset{}{^*}\sum_{l_r=N-r+2}^{\infty} \sqrt{l_r+1/2}\, \varphi_{l_r}(t) \sum_{k=l_r+r}^{\infty} \sqrt{k+1/2}\langle f, \varphi_k\rangle B_k^r,$$

where $B_k^1 = 1$ and for r= 2, 3,...

$$B_k^r := \sideset{}{^*}\sum_{l_1=l_r+r-1}^{k-1} (l_1+1/2) \sideset{}{^*}\sum_{l_2=l_r+r-2}^{l_1-1} (l_2+1/2) \ldots \sideset{}{^*}\sum_{l_{r-2}=l_r+2}^{l_{r-3}-1} (l_{r-2}+1/2) \sideset{}{^*}\sum_{l_{r-1}=l_r+1}^{l_{r-2}-1} (l_{r-1}+1/2).$$

Easy to see that

$$B_k^r = c^* k^{2r-2}, \tag{3.1}$$

where the factor c^* does not depend on N, k.

At first, we consider the case $1 < s < \infty$. Using Hölder inequality and the definition of W_s^μ, for $\mu > 2r - 1/s + 1/2$ we get

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f\|_2^2 = 2^{2r} \sideset{}{^*}\sum_{l_r=0}^{N-r+1} (l_r+1/2) \left(\sum_{k=N+1}^{\infty} \sqrt{k+1/2}\, \langle f, \varphi_k\rangle B_k^r \right)^2$$

$$+2^{2r} \sum_{l_r=N-r+2}^{\infty}{}^{*} (l_r+1/2) \left(\sum_{k=l_r+r}^{\infty} \sqrt{k+1/2}\, \langle f, \varphi_k \rangle B_k^r \right)^2$$

$$\leq c \sum_{l_r=0}^{N-r+1} (l_r+1/2) \left(\sum_{k=N+1}^{\infty} k^{s\mu} \left|\langle f, \varphi_k\rangle\right|^s \right)^{2/s} \left(\sum_{k=N+1}^{\infty} k^{(2r-\mu-3/2)\frac{s}{s-1}} \right)^{2(s-1)/s}$$

$$+c \sum_{l_r=N-r+2}^{\infty} (l_r+1/2) \left(\sum_{k=l_r+r}^{\infty} k^{s\mu} \left|\langle f, T_k\rangle\right|^s \right)^{2/s} \left(\sum_{k=l_r+r}^{\infty} k^{(2r-\mu-3/2)\frac{s}{s-1}} \right)^{2(s-1)/s}$$

$$\leq c\|f\|_{s,\mu}^2 N^{-2(\mu-2r+3/2)+\frac{2(s-1)}{s}+2}.$$

Thus, we find

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f\|_2 \leq c\|f\|_{s,\mu} N^{-\mu+2r-1/s+1/2}.$$

In the case of $s=1$ the assertion of Lemma is proved similarly. □

Lemma 3.2. *Let the condition (1.2) be satisfied for $1 \leq p \leq \infty$. Then for arbitrary function $f \in L_2$ it holds*

$$\|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_2 \leq c\delta N^{2r-1/p+1/2}.$$

Proof. We take the representation (2.4) and change the order of summation

$$\mathcal{D}_N^{(r)} f(t) - \mathcal{D}_N^{(r)} f^\delta(t)$$

$$= 2^r \sum_{k=r}^{N} \sqrt{k+1/2}\langle f-f^\delta, \varphi_k\rangle \sum_{l_1=r-1}^{k-1}{}^{*} (l_1+1/2) \sum_{l_2=r-2}^{l_1-1}{}^{*} (l_2+1/2) \cdots \sum_{l_r=0}^{l_{r-1}-1}{}^{*} \sqrt{l_r+1/2}\varphi_{l_r}(t)$$

$$= 2^r \sum_{k=r}^{N} \sqrt{k+1/2}\langle f-f^\delta, \varphi_k\rangle \ldots \sum_{l_{r-2}=2}^{l_{r-3}-1}{}^{*} (l_{r-2}+1/2) \sum_{l_r=0}^{l_{r-2}-2}{}^{*} \sqrt{l_r+1/2}\, \varphi_{l_r}(t) \sum_{l_{r-1}=l_r+1}^{l_{r-2}-1}{}^{*} (l_{r-1}+1/2)$$

$$= \cdots = 2^r \sum_{k=r}^{N} \sqrt{k+1/2}\langle f-f^\delta, \varphi_k\rangle \sum_{l_1=r-1}^{k-1}{}^{*} (l_1+1/2) \sum_{l_r=0}^{l_1-r+1}{}^{*} \sqrt{l_r+1/2}\varphi_{l_r}(t) \sum_{l_2=l_r+r-2}^{l_1-1}{}^{*} (l_2+1/2)$$
$$\ldots \sum_{l_{r-2}=l_r+2}^{l_{r-3}-1}{}^{*} (l_{r-2}+1/2) \sum_{l_{r-1}=l_r+1}^{l_{r-2}-1}{}^{*} (l_{r-1}+1/2)$$

$$= 2^r \sum_{k=r}^{N} \sqrt{k+1/2}\langle f-f^\delta, \varphi_k\rangle \sum_{l_r=0}^{k-r}{}^{*} \sqrt{l_r+1/2}\, \varphi_{l_r}(t) \sum_{l_1=l_r+r-1}^{k-1}{}^{*} (l_1+1/2) \sum_{l_2=l_r+r-2}^{l_1-1}{}^{*} (l_2+1/2)$$
$$\ldots \sum_{l_{r-2}=l_r+2}^{l_{r-3}-1}{}^{*} (l_{r-2}+1/2) \sum_{l_{r-1}=l_r+1}^{l_{r-2}-1}{}^{*} (l_{r-1}+1/2).$$

As a result we have

$$\mathcal{D}_N^{(r)} f(t) - \mathcal{D}_N^{(r)} f^\delta(t) = 2^r \sum_{l_r=0}^{N-r}{}^{*} \sqrt{l_r+1/2}\, \varphi_{l_r}(t) \sum_{k=l_r+r}^{N} \sqrt{k+1/2}\, \langle f-f^\delta, \varphi_k\rangle B_k^r.$$

At first, we consider the case $1 < p < \infty$. Using Hölder inequality, we get

$$\|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_2^2 \le 2^{2r} \sum_{l_r=0}^{N-r} (l_r + 1/2) \left(\sum_{k=l_r+r}^{N} \sqrt{k+1/2}\ |\langle f - f^\delta, \varphi_k\rangle| B_k^r \right)^2$$

$$\le c \sum_{l_r=0}^{N-r} (l_r + 1/2) \left(\sum_{k=l_r+r}^{N} |\xi_k|^p \right)^{2/p} \left(\sum_{k=l_r+r}^{N} k^{(2r-3/2)\frac{p}{p-1}} \right)^{2(p-1)/p}$$

$$\le c\, \delta^2 N^{2(2r-3/2)+2(p-1)/p+2}.$$

In the case of $p = 1$ the assertion of Lemma is proved similarly. □

The combination of Lemmas 3.1 and 3.2 gives

Theorem 3.3. *Let $f \in W_s^\mu$, $1 \le s < \infty$, $\mu > 2r - 1/s + 1/2$, and let the condition (1.2) be satisfied for $1 \le p \le \infty$. Then for $N \asymp \delta^{-\frac{1}{\mu-1/p+1/s}}$ it holds*

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f^\delta\|_2 \le c\delta^{\frac{\mu-2r+1/s-1/2}{\mu-1/p+1/s}}.$$

Corollary 3.4. *In the considered problem, the truncation method $\mathcal{D}_N^{(r)}$ (2.1) achieves the accuracy*

$$O\left(\delta^{\frac{\mu-2r+1/s-1/2}{\mu-1/p+1/s}}\right)$$

on the class W_s^μ, $\mu > 2r - 1/s + 1/2$, and requires

$$\mathrm{card}([r, N]) \asymp N \asymp \delta^{-\frac{1}{\mu-1/p+1/s}}$$

perturbed Fourier-Legendre coefficients.

Remark 3.5. *In [16] it was shown that on the Wiener classes, Chebyshev polynomials guarantee in the C-metric a higher accuracy of numerical differentiation than the Legendre polynomials. However, the Wiener class that uses Fourier-Legendre coefficients is broader than the corresponding Wiener class that uses Chebyshev coefficients. Therefore, it is not surprising that higher accuracy can be achieved on narrower classes. Furthermore, a comparison of our results with those in [16] shows that in the L_2-metric, Legendre and Chebyshev polynomials provide the same order of accuracy.*

4 Error estimate in the L_q–metrics

In the preceding sections, we have established upper bounds for the accuracy of the truncation method $\mathcal{D}_N^{(r)}$ (2.1) when measured in the C and L_2 metrics. Now, we extend our analysis to derive comparable bounds across the complete spectrum of spaces L_q for $2 \le q \le \infty$.

To facilitate this extension, we introduce several key facts and definitions. A well-established result (see, for example, Theorem 6.6 [5]) states that for any $2 \le q \le \infty$ and any algebraic polynomial P_N of degree N

$$\|P_N\|_q \le c\, N^{(1-\frac{2}{q})}\, \|P_N\|_2. \tag{4.1}$$

Let us define the best approximation quantity as

$$\mathbb{E}_N(f)_2 = \inf_{P \in \Pi_N} \|f - P\|_2,$$

where Π_N represents the collection of polynomials with degree $\le N$.

Theorem 11.1 in [5] establishes that for any function $g \in L_2$ and $2 < q \leq \infty$, the following inequality holds

$$\|g\|_q \leq c \left[\left\{ \sum_{k=1}^{\infty} k^{(1-\frac{2}{q})q_1 - 1} \, \mathbb{E}_k(g)_2^{q_1} \right\}^{1/q_1} + \|g\|_2 \right], \tag{4.2}$$

where q_1 is defined as

$$q_1 = \begin{cases} q, & \text{if } q < \infty, \\ 1, & \text{if } q = \infty. \end{cases}$$

We now present an upper estimate for the first difference term from the right-hand side of equation (2.3) in the L_q metrics.

Lemma 4.1. *Let $f \in W_s^{\mu}$, $1 \leq s < \infty$, $\mu > 2r - 1/s - 2/q + 3/2$, $2 \leq q \leq \infty$. Then the following bound holds*

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f\|_q \leq c \, N^{-\mu + 2r - 1/s - 2/q + 3/2}.$$

Proof. Let us define $g = f^{(r)} - \mathcal{D}_N^{(r)} f$. According to Lemma 3.1, we have

$$\|g\|_2 = \|f^{(r)} - \mathcal{D}_N^{(r)} f\|_2 \leq c \, \|f\|_{s,\mu} N^{-\mu + 2r - 1/s + 1/2}. \tag{4.3}$$

First, consider the case $2 < q < \infty$. From (4.2), we derive

$$\|g\|_q \leq c \left[\left\{ \sum_{k=1}^{\infty} k^{q-3} \, \mathbb{E}_k(g)_2^{q} \right\}^{1/q} + \|g\|_2 \right]. \tag{4.4}$$

We separate this summation by k into two components:

$$S_1 = \sum_{k=1}^{N-r} k^{q-3} \, \mathbb{E}_k(g)_2^q,$$

$$S_2 = \sum_{k=N-r+1}^{\infty} k^{q-3} \, \mathbb{E}_k(g)_2^q.$$

For the polynomial P in $\mathbb{E}_k(g)_2$, we select

$$P = \begin{cases} 0, & \text{if } 1 \leq k \leq N - r, \\ \mathcal{D}_k^{(r)} f - \mathcal{D}_N^{(r)} f, & \text{if } k \geq N - r + 1. \end{cases} \tag{4.5}$$

Using relation (4.4), we can bound S_1 as follows. Since we choose $P = 0$ for $1 \leq k \leq N - r$ in the definition of $\mathbb{E}_k(g)_2$, we have

$$S_1 = \sum_{k=1}^{N-r} k^{q-3} \, \mathbb{E}_k(g)_2^q \leq \sum_{k=1}^{N-r} k^{q-3} \, \|g - 0\|_2^q = \sum_{k=1}^{N-r} k^{q-3} \, \|f^{(r)} - \mathcal{D}_N^{(r)} f\|_2^q$$

$$\leq c \, N^{-(\mu - 2r + 1/s - 1/2)q} \sum_{k=1}^{N-r} k^{q-3} \leq c \, N^{-(\mu - 2r + 1/s + 2/q - 3/2)q}.$$

For $\mu > 2r - 1/s - 2/q + 3/2$, we can bound S_2 as

$$S_2 \leq \sum_{k=N-r+1}^{\infty} k^{q-3} \, \|f^{(r)} - \mathcal{D}_k^{(r)} f\|_2^q$$

$$\le c \sum_{k=N-r+1}^{\infty} k^{-(\mu-2r+1/s+2/q-3/2)q-1} \le c\, N^{-(\mu-2r+1/s+2/q-3/2)q}.$$

Consequently, we have

$$(S_1 + S_2)^{1/q} \le c\, N^{-\mu+2r-1/s-2/q+3/2}. \tag{4.6}$$

Substituting (4.3) and (4.6) into relation (4.4), we obtain

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f\|_q \le c\,(N^{-\mu+2r-1/s-2/q+3/2} + N^{-\mu+2r-1/s+1/2}) = c\, N^{-\mu+2r-1/s-2/q+3/2}.$$

Now consider the case where $q = \infty$. From inequality (4.2), we have

$$\|g\|_\infty \le c\left(\bar{S}_1 + \bar{S}_2 + \|g\|_2\right), \tag{4.7}$$

where

$$\bar{S}_1 = \sum_{k=1}^{N-r} \mathbb{E}_k(g)_2,$$

$$\bar{S}_2 = \sum_{k=N-r+1}^{\infty} \mathbb{E}_k(g)_2.$$

As before, we use P from (4.5) in $\mathbb{E}_k(g)_2$. Using estimation (4.3), we derive

$$\bar{S}_1 \le \sum_{k=1}^{N-r} \|f^{(r)} - \mathcal{D}_N^{(r)} f\|_2$$

$$\le c\, N^{-\mu+2r-1/s+1/2} \sum_{k=1}^{N-r} 1 = c\, N^{-\mu+2r-1/s+3/2}.$$

For $\mu > 2r - 1/s + 3/2$, we have

$$\bar{S}_2 \le \sum_{k=N-r+1}^{\infty} \|f^{(r)} - \mathcal{D}_k^{(r)} f\|_2$$

$$\le c \sum_{k=N-r+1}^{\infty} k^{-\mu+2r-1/s+1/2} = c\, N^{-\mu+2r-1/s+3/2}.$$

This yields

$$\bar{S}_1 + \bar{S}_2 \le c\, N^{-\mu+2r-1/s+3/2}. \tag{4.8}$$

Substituting (4.3) and (4.8) into inequality (4.7), we obtain

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f\|_\infty \le c\, N^{-\mu+2r-1/s+3/2}.$$

For the case $q = 2$, the result was established in Lemma 3.1.
Thus, Lemma is fully proven for all applicable values of q. □

Next, we provide an estimate for the second difference term from the right-hand side of (2.3) in the L_q-metric.

Lemma 4.2. *Let the condition (1.2) be satisfied for $1 \le p \le \infty$. Then for any function $f \in L_2$ and $2 \le q \le \infty$, the following bound holds*

$$\|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_q \le c\,\delta\, N^{2r-1/p-2/q+3/2}.$$

Proof. From Lemma 3.2, we know

$$\|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_2 \le c\,\delta\, N^{2r-1/p+1/2}.$$

Applying inequality (4.1) to the polynomial $\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta$ of degree at most N, we obtain for any $2 \le q \le \infty$

$$\|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_q \le c\, N^{(1-2/q)} \|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_2$$

$$\le c\, N^{(1-2/q)} \cdot c\,\delta\, N^{2r-1/p+1/2} = c\,\delta\, N^{2r-1/p-2/q+3/2},$$

which completes the proof. $\square$

By combining Lemmas 4.1 and 4.2, we arrive at the following statement.

Theorem 4.3. *Let $f \in W_s^\mu$, $1 \le s < \infty$, $\mu > 2r - 1/s - 2/q + 3/2$, $2 \le q \le \infty$, and let the condition (1.2) be satisfied for $1 \le p \le \infty$. Then for $N \asymp \delta^{-\frac{1}{\mu-1/p+1/s}}$ it holds*

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f^\delta\|_q \le c\,\delta^{\frac{\mu-2r+1/s+2/q-3/2}{\mu-1/p+1/s}}.$$

Proof. Applying Lemmas 4.1 and 4.2 to equation (2.3), we obtain

$$\|f^{(r)} - \mathcal{D}_N^{(r)} f^\delta\|_q \le \|f^{(r)} - \mathcal{D}_N^{(r)} f\|_q + \|\mathcal{D}_N^{(r)} f - \mathcal{D}_N^{(r)} f^\delta\|_q$$

$$\le c\, N^{-\mu+2r-1/s-2/q+3/2} + c\,\delta N^{2r-1/p-2/q+3/2}$$

$$= c\, N^{2r-2/q+3/2} \left(N^{-\mu-1/s} + \delta N^{-1/p}\right).$$

By substituting $N \asymp \delta^{-\frac{1}{\mu-1/p+1/s}}$ into this expression, we complete the proof of Theorem. $\square$

Corollary 4.4. *For the problem under consideration, the truncation method $\mathcal{D}_N^{(r)}$ (2.1) achieves an accuracy of*

$$O\left(\delta^{\frac{\mu-2r+1/s+2/q-3/2}{\mu-1/p+1/s}}\right)$$

on the class W_s^μ, $\mu > 2r - 1/s - 2/q + 3/2$, requiring

$$\mathrm{card}([r, N]) \asymp N \asymp \delta^{-\frac{1}{\mu-1/p+1/s}}$$

perturbed Fourier-Legendre coefficients.

Remark 4.5. *In previous research (see [9]), error estimates for the truncation method $\mathcal{D}_N^{(r)}$ (2.1) were only established for the specific case where $r = 1$, $p = s = 2$, with output spaces limited to C and L_2. Thus, Theorems 2.3 and 4.3 significantly expand upon these earlier findings by generalizing to arbitrary values of r, p, s, and q.*

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