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Control problem for the Markov-modulated Poisson process in the averaging schema

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Анотація. У цій статті розглядається задача керування стохастичною еволюційною системою під впливом марковсько-модульованих пуассонівських збурень у схемі усереднення. Такий процес збурення поєднує пуассонівський процес з марковським процесом, який модулює інтенсивність стрибків. Ця постановка задачі дозволяє моделювати системи з переходами між режимами або зі значними рідкісними стрибками. У цій задачі функція керування визначається умовою екстремуму функції критерію якості. В першу чергу досліджуються асимптотичні властивості стохастичного процесу еволюції та керування. Для цього побудовано генератор для граничного процесу, що дає змогу розв'язати задачу про сингулярне збурення вихідного процесу. Отриманий результат дозволяє сформулювати і довести теорему про слабку збіжність еволюційного процесу до усередненого. Розв'язок детермінованого диференціального рівняння визначає граничний процес для стохастичної еволюційної системи на зростаючих часових інтервалах. Отриманий результат дозволяє сформулювати умови, які забезпечують слабку збіжність функції керування до оптимального керування.

Abstract. This paper discusses the control problem for the stochastic evolution system under Markov-modulated Poisson perturbations in the averaging schema. Such a perturbation process combines a Poisson process with a Markov process that modulates the intensity of jumps. Formulating the problem allows us to model systems with transitions between regimes or significant rare jumps. In this problem, the control function is determined by the condition of the extremum of the quality criterion function. The asymptotic properties of the stochastic evolution and control coupled process are being investigated initially. For this purpose, we build the generator for the limit process, solving the singular perturbation problem of the original process. The obtained result allows us to formulate and prove the theorem about weak convergence of the evolution process to the averaged one. The solution of a deterministic differential equation determines the limit process for a stochastic evolution system at increasing time intervals. Then, the obtained result allows us to formulate the conditions that ensure the weak convergence of the control function to optimal control.

1 Introduction

The study of stochastic evolutionary systems is becoming increasingly important in various fields of science, including network traffic modeling, financial modeling [4], and modeling of service and reliability since these systems can effectively describe the randomness and uncertainty inherent in

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real-world phenomena. This paper investigates the control problem for a stochastic evolution system influenced by Markov-modulated Poisson [5] perturbations in the averaging schema. Such a type of disturbance combines a Poisson process with a Markov process that modulates the intensity of jumps. This allows us to model systems with transitions between regimes or significant rare jumps, often observed in natural and technical processes. Our primary objective is to analyze the asymptotic behavior of the controlled stochastic evolution system and establish conditions for its convergence to an averaged deterministic counterpart.

A key aspect of our investigation is formulating an optimal control function based on a stochastic optimization procedure. By defining the control function through an extremum condition on the quality criterion, we derive sufficient conditions under which the optimization procedure converges to an optimal control with probability one. These conditions rely on the stability properties of the averaged system and appropriate constraints on the optimization scheme's learning rate and step size.

2 Problem statement

We consider the optimal control problem for a stochastic evolution system influenced by the Markov-modulated Poisson perturbations. The system dynamics are defined by a stochastic differential equation in which both the control function and the stochastic disturbances are subject to the influence of an underlying ergodic Markov process. The task is to determine a control strategy $u^\varepsilon(t)$ that minimizes quality criterion $J(y, u, x)$. To achieve this, we consider the averaging schema approach, which allows us to approximate the behavior of the original perturbed system by an averaged deterministic counterpart in the limit. The aim is to analyze the asymptotic behavior of the coupled stochastic system and optimization procedure, establish the weak convergence of the controlled process, and derive sufficient conditions under which the control function converges to an optimal solution with probability one.

Let's consider a stochastic evolution system in an ergodic Markov environment given by the evolution equation [10, p.54]

$$dy^\varepsilon(t) = C(y^\varepsilon(t), u^\varepsilon(t), x(t/\varepsilon))dt + a(t)\tilde{N}_x(t)dt, \tag{2.1}$$

where $y^\varepsilon(t) \in D(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ is Skorokhod space of càdlàg functions [2, p.123] (right-continuous with left limits) on $\mathbb{R}_{\geq 0}$ with values in $\mathbb{R}^n$. $a(t) \in C(\mathbb{R}_{\geq 0})$ represents the size and effect of the jump at time t and $u^\varepsilon(t) \in C(\mathbb{R}_{\geq 0})$ is a control function.

An uniformly ergodic Markov process [6, p.6] $x(t), t \geq 0$ in the standard phase space $(X, \mathbf{X})$ is defined by the generator [7, p.7]

$$\mathbf{Q}\varphi(x) = q(x) \int_X P(x, dy) \left[\varphi(y) - \varphi(x)\right], \varphi \in \mathbf{B}(X).$$

Here $\mathbf{B}(X)$ is a Banach space of bounded functions with supremum-norm $||\varphi|| = \max_{x \in X} |\varphi(x)|$ [7, p.2].

Stochastic kernel $P(x, B), x \in X, B \in \mathbf{X}$ defines uniformly ergodic embedded Markov chain $x_n = x(\tau_n)$, $n \geq 0$, with stationary distribution $\rho(B), B \in \mathbf{X}$. Stationary distribution $\pi(B), B \in \mathbf{X}$ of the Markov process $x(t), t \geq 0$, is defined by the representation

$$\pi(dx)q(x) = \hat{q}\rho(dx), \ \hat{q} = \int_X \pi(dx)q(x).$$

We denote $\mathbf{R_0}$ as a potential operator of the generator $\mathbf{Q}$, which is defined by $\mathbf{R_0} = \mathbf{\Pi} - (\mathbf{\Pi} + \mathbf{Q})^{-1}$, where $\mathbf{\Pi}\varphi(x) = \int_X \pi(dy)\varphi(y)$ is a projector on the zeros subspace $Z_Q = \{\varphi : \mathbf{Q}\varphi = 0\}$ of the operator $\mathbf{Q}$.

Poisson process $N_x^\varepsilon(t)$ is the process of counting the number of events that have occurred up to time t. The arrival rate of events at the moment t is defined as $\lambda(x(t/\varepsilon))$; that is, the intensity parameter is modulated by the Markov process of the state [9, p.10].

Compensated Poisson process $\tilde{N}_x(t)$ defined as

$$\tilde{N}_x(t) = N_x^\varepsilon(t) - \Lambda(t) = N_x^\varepsilon(t) - \int_0^t \lambda(x(s))ds. \tag{2.2}$$

The compensated Poisson process is used to center the jumps, i.e., $E[\tilde{N}_x(t)] = 0$.

It can be shown [12, p.104] that the generator of the process $\tilde{N}_x(t)$ (2.2) has the form

$$\tilde{\mathbf{W}}\varphi(N) = \lambda(x)[\mathbf{R}_+ - \mathbf{I}]\varphi(N) - \lambda(x)\varphi'(N),$$

where $\mathbf{R}_+\varphi(N) = \varphi(N+1)$ and $\mathbf{I}\varphi(N) = \varphi(N)$.

Also, it can be shown that the averaging of the process $\tilde{N}_x(t)$ along the stationary distribution of the moduling Markov process equals zero, i.e

$$\mathbf{\Pi}\tilde{N}_x(t) = \int_X \pi(dx)\tilde{N}_x(t) = 0. \tag{2.3}$$

3 Stochastic optimization procedure for control function

Control function [1, p.1] $u^\varepsilon(t)$ for equation (2.1) is defined by the quality criterion $J(y, u, x)$, that has unique extremum (we will consider Infinium for the sake of simplicity) for each value of process y and for each state x of the Markov process $x(t)$ at interval $[t_i, t_{i+1})$. More precisely

$$u^* = arg \inf_u J(y, u, x).$$

Lets also $J(\cdot, u, \cdot) \in C^2(\mathbb{R}^n)$, i.e. twice continuously differentiable.

Control $u(t)$ is completely determined by the system

$$\frac{\partial J(y, u, x)}{\partial u_i} = 0, i = \overline{0, n}.$$

Consider stochastic optimization procedure for control function $u^\varepsilon(t)$ in the form

$$du^\varepsilon(t) = \alpha(t)\nabla_{\beta(t)}J(y^\varepsilon(t), u^\varepsilon(t), x(t/\varepsilon))dt, \tag{3.1}$$

where $\alpha(t)$ is the learning rate and $\beta(t)$ is a difference scheme step.

$$\nabla_{\beta(t)}J(\cdot, u, \cdot) = \frac{J(\cdot, u_i^+, \cdot) - J(\cdot, u_i^-, \cdot)}{2\beta(t)}, i = \overline{0, n}$$
$$u_i^\pm = u \pm \beta(t)e_i, e_i = (0, \ldots, 1, \ldots, 0), i = \overline{0, n}.$$

Functions $\alpha(t)$ and $\beta(t)$ are satisfying conditions $\alpha(t) \to 0$, $\beta(t) \to 0$ for $t \to 0$.

Let general initial conditions have the form

$$y(0) = y_0, u(0) = u_0, x(0) = x_0. \tag{3.2}$$

Lemma 3.1. *Generator of the four-component process $(y^\varepsilon(t), u^\varepsilon(t), \tilde{N}_x(t), x(t/\varepsilon))$ has the asymptotical representation [8]*

$$\begin{aligned} \mathbf{L}^\varepsilon(x)\varphi(y, u, w, x) &= \varepsilon^{-1}\mathbf{Q}\varphi(y, u, w, x) + \mathbf{B_t}(x)\varphi(y, u, w, x) \\ &+ (\mathbf{C}(u, x) + \mathbf{A_t})\varphi(y, u, w, x) + \mathbf{W}(x)\varphi(y, u, w, x). \end{aligned} \tag{3.3}$$

where

$$\mathbf{B_t}(x)\varphi(y,u,w,x) = \alpha(t)\nabla_{\beta(t)}J(y,u,x)\varphi'_u(y,u,w,x),$$
$$\mathbf{C}(u,x)\varphi(y,u,w,x) = C(y,u,x)\varphi'_y(y,u,w,x),$$
$$\mathbf{A_t}\varphi(y,u,w,x) = a(t)w\varphi'_y(y,u,w,x),$$
$$\mathbf{W}(x)\varphi(y,u,w,x) = \lambda(x)[\mathbf{R}_+ - \mathbf{I}]\varphi(y,u,w,x)-$$
$$-\lambda(x)\varphi'_w(y,u,w,x).$$

Proof. Similar to lemma 1 in [12, p.103], we calculate the conditional expectation within the definition of the process generator. □

Lemma 3.2. *Singular perturbation problem [13] for the operator* (3.3) *on the test functions*

$$\varphi^\varepsilon(y,u,w,x) = \varphi(y,u,w) + \varepsilon\varphi_0(y,u,w,x)$$

has the solution in the form

$$\mathbf{L}^\varepsilon(x)\varphi^\varepsilon(y,u,w,x) = \mathbf{L}\varphi(y,u,w) + \varepsilon\theta(x)\varphi(y,u,w,x), \tag{3.4}$$

where the remaining term $\theta^\varepsilon(x)$ is uniformly bounded on x.

Limit operator $\mathbf{L}$ *is defined by*

$$\mathbf{L\Pi} = \mathbf{\Pi B_t \Pi} + \mathbf{\Pi C}(u,x)\mathbf{\Pi} + \mathbf{\Pi W}(x)\mathbf{\Pi}. \tag{3.5}$$

Proof. Lets conduct the similar terms with respect to ε to proof the equality (3.4)

$$\mathbf{L}^\varepsilon(x)\varphi^\varepsilon(y,u,w,x) = \varepsilon^{-1}\mathbf{Q}\varphi(y,u,w)+$$
$$\mathbf{Q}\varphi_0(y,u,w,x) + [\mathbf{B_t}(x) + \mathbf{C}(u,x) + \mathbf{A_t} + \mathbf{W}(x)]\varphi(y,u,w)$$
$$+\varepsilon[\mathbf{B_t}(x) + \mathbf{C}(u,x) + \mathbf{A_t} + \mathbf{W}(x)]\varphi_0(u,w,x).$$

Since $\varphi(y,u,w)$ doesn't depend on x, then

$$\mathbf{Q}\varphi(y,u,w) = 0, \Leftrightarrow \varphi(y,u,w) \in Z_Q.$$

The following term would be written in the form $\mathbf{Q}\varphi_0(y,u,w,x) + [\mathbf{B_t} + \mathbf{C}(u,x) + \mathbf{A_t} + \mathbf{W}(x)]\varphi(y,u,w) = \mathbf{L}\varphi(y,u,w)$.

According to (2.3): $\mathbf{\Pi A_t} = \mathbf{\Pi}a(t)\tilde{N}_x(t) = a(t)\mathbf{\Pi}\tilde{N}_x(t) = 0$. So we can obtain limit process $\mathbf{L}$ in the form ((3.5)) using the solution condition of the last equation. Then

$$\varphi_0(y,u,w,x) = \mathbf{R_0}[\mathbf{B_t}(x) + \mathbf{C}(u,x) + \mathbf{A_t} + \mathbf{W}(x) - \mathbf{L}]\varphi(y,u,w)$$

and taking into account that $\mathbf{R_0 L} = 0$, we obtain

$$\varphi_0(y,u,w,x) = \mathbf{R_0}[\mathbf{B_t}(x) + \mathbf{C}(u,x) + \mathbf{A_t} + \mathbf{W}(x)]\varphi(y,u,w). \tag{3.6}$$

Using (3.6) we can bring the last term to the form

$$[\mathbf{B_t} + \mathbf{C}(u,x) + \mathbf{A_t} + \mathbf{W}(x)]\varphi_0(y,u,w,x) =$$
$$\varepsilon[\mathbf{B_t}(x) + \mathbf{C}(u,x) + \mathbf{A_t} + \mathbf{W}(x)]$$
$$\mathbf{R_0}[\mathbf{B_t}(x) + \mathbf{C}(u,x) + \mathbf{A_t} + \mathbf{W}(x)]\varphi(y,u,w) =$$
$$\varepsilon\theta(x)\varphi(y,u,w).$$

We can proof that $\theta(x)$ on the functions $\varphi(y,u,w)$ is bounded using the form of operators $\mathbf{B_t}(x)$, $\mathbf{C}(u,x)$, $\mathbf{A_t}$, $\mathbf{R}_+$, $\mathbf{W}(x)$ and $\mathbf{R_0}$. □

Theorem 3.3. *The weak convergence holds true*

$$(y^{\varepsilon}(t), u^{\varepsilon}(t), N_x^{\varepsilon}(t)) \to (\hat{y}(t), \hat{u}(t), \hat{N}(t)), \quad \varepsilon \to 0.$$

Limit process $(\hat{y}(t), \hat{u}(t), \hat{N}(t))$ *is defined by the generator*

$$\mathbf{L}\varphi(y,u,w) = \hat{\mathbf{C}}(y,u)\varphi(y,u,w) + \mathbf{W}\varphi(y,u,w) + \hat{\mathbf{B}}_{\mathbf{t}}\varphi(y,u,w), \tag{3.7}$$

where

$$\begin{gathered}
\hat{\mathbf{C}}(y,u)\varphi(y,u,w) = \hat{C}(y,u)\varphi'_y(y,u,w),\\
\hat{C}(y,u) = \mathbf{\Pi}C(u,x) = \int_X \pi(dx)C(y,u,x),\\
\mathbf{W}\varphi(y,u,w) = \Lambda_{\Pi}\mathbf{\Pi}(\mathbf{R}_{+} - \mathbf{I})\varphi(y,u,w) - \Lambda_{\Pi}\varphi'_w(y,u,w),\\
\Lambda_{\Pi} = \mathbf{\Pi}\lambda(x) = \int_X \pi(dx)\lambda(x),\\
\hat{\mathbf{B}}_{\mathbf{t}}\varphi(y,u,w) = \alpha(t)\nabla_{\beta(t)}J(y,u)\varphi'_u(y,u,w),\\
J(y,u) = \mathbf{\Pi}J(y,u,x) = \int_X \pi(dx)J(y,u,x).
\end{gathered}$$

Proof. After calculating the right part of the (3.5) we obtain the generator $\mathbf{L}$ in form (3.7). Theorem proof completion is made using the theorem 3.2 from [7, p.79].

In this case limit process $(\hat{y}(t), \hat{u}(t))$ for control problem is defined by the differential equations

$$\begin{gathered}
d\hat{y}(t) = \hat{C}(\hat{y}(t), \hat{u}(t))dt,\\
d\hat{u}(t) = \alpha(t)\nabla_{\beta(t)}J(\hat{y}(t), \hat{u}(t))dt.
\end{gathered}$$

□

Theorem 3.4. *Let Lyapunov function* $V(y,u)$ *of the averaged system*

$$\frac{du}{dt} = J^*(y,u),$$

where $J^*(\cdot,u) = \nabla J(\cdot,u) = \{\frac{\partial J}{\partial u_i}, i = \overline{0,n}\}$ *for arbitrary value of the process* y*, exists and provides its exponential stability*

$$J^*(y,u)V'_u(y,u) \le -c_0 V(y,u), c_0 > 0. \tag{3.8}$$

Let for $\beta(t) > 0$ *function* $J(y,u)$ *satisfy global Lipshiz conditions*

$$|\nabla_{\beta(t)}J(y,u) - J^*(y,u)| \le c_1\beta(t), c_1 > 0. \tag{3.9}$$

Let also

$$\tag{3.10}$$

and

$$\int_{t_0}^{\infty} \alpha(t)dt = \infty, \int_{t_0}^{\infty} \alpha(t)\beta(t)dt < \infty, t_0 > 0. \tag{3.11}$$

Then the solution for control problem (2.1), (3.1), (3.2) *converges with probability 1 to the optimal control of the averaged system, i.e.*

$$P\{\lim_{\varepsilon\to 0} u^{\varepsilon}(t) = u(t)\} = 1.$$

Lemma 3.5. *Generator of the control function* $\mathbf{B_t}(x)$ *allows an asymptotic representation*

$$\mathbf{B_t}(x)V^e(y,u,w,x) = \hat{\mathbf{B}}_\mathbf{t}V(y,u) + \varepsilon\theta_V(x)V(y,u),$$

where $||\theta_V(x)V(y,u)|| \leq M, M > 0$, $\hat{\mathbf{B}}_\mathbf{t}V(y,u) = \alpha(t)\nabla_{\beta(t)}J(y,u)V'_u(y,u)$ *and*

$$V^e(y,u,w,x) = V(y,u) + \varepsilon V_1(y,u,w,x)$$

is a perturbed Lyapunov function.

Proof. From lemma 3.5 and conditions (3.8), (3.9) and (3.10) the following estimation can be obtained

$$\mathbf{B_t}(x)V^e(y,u,w,x) \leq -c_0\alpha(t)V(y,u) + c^*\alpha(t)\beta(t)(1 + V(y,u)).$$

Then the theorem proof completion is made using the condition (3.11) similar to the Theorem 2 in [3, p.111]. □

4 Conclusions

The obtained result allows us to construct the limit process and the stochastic optimization procedure for the initial control problem for stochastic evolution with the Markov-modulated Poisson perturbations and control, determined by the condition for the extremum of the quality criterion function. Sufficient conditions for the convergence of the procedure to optimal control with probability one are obtained for the averaging schema. Further work may include investigation of the fluctuations of the evolution system [11, p.717] while optimal control would be applied.

Declarations

Conflict of Interest.
The authors have no conflict of interest to declare that are relevant to the content of this article.
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