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# On the numerical solution of the Cauchy problem for the Klein–Gordon equation using the Landweber method based on integral equations

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Анотація. У цій роботі розглянуто наближене розв'язування задачі Коші для рівняння Клейна-Гордона у двовимірній двозв'язній області. Через некоректний характер задачі застосовано ітераційний метод регуляризації Ландвебера. Кожна ітерація включає розв'язування двох коректно поставлених мішаних крайових задач Діріхле-Неймана. До кожної із цих задач застосовано метод граничних інтегральних рівнянь з використанням потенціалів та отримано систему інтегральних рівнянь другого роду, яку розв'язано за допомогою методу Нистрьома після параметризації та з врахуванням особливостей в ядрах. Чисельні експерименти для задачі Коші наведено для ілюстрації ефективності та точності запропонованого підходу.

Abstract. This work addresses the numerical solution of the Cauchy problem for the Klein-Gordon equation in a two-dimensional doubly connected domain. Due to the ill-posed nature of the problem, the Landweber iterative regularization method is employed. Each iteration involves solving two well-posed mixed Dirichlet–Neumann boundary value problems. For each of these problems, the boundary integral equation method is applied using potential theory, resulting in a system of second-kind integral equations. This system is solved using the Nyström method after parameterization and with appropriate handling of the kernel singularities. Numerical experiments for the Cauchy problem are provided to demonstrate the effectiveness and accuracy of the proposed approach.

## 1 Introduction

Let $D_2 \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial D_2 = \Gamma_2$. Let $D_1 \subset \mathbb{R}^2$ be a domain with boundary $\partial D = \Gamma_1$, such, that $\overline{D}_1 \subset D_2$. Define the domain $D$ as $D = D_2 \backslash \overline{D}_1$ with boundary $\partial D = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \in C^3$, $\Gamma_2 \in C^2$. We denote by $\nu$ the outward-pointing unit normal vector to the boundary (see Fig. 1.1).

We consider the Cauchy problem for the Klein-Gordon equation. The aim is to find a function $u \in C^2(D) \cap C^1(\overline{D})$, that satisfies the following partial differential equation

$$\Delta u - \kappa^2 u = 0 \quad \text{in } D, \tag{1.1}$$

where $\kappa > 0$ is a given constant.

The Cauchy data are prescribed on the boundary $\Gamma_2 \subset \partial D$ in the form

$$u = f_1, \quad \frac{\partial u}{\partial \nu} = f_2 \quad \text{on } \Gamma_2, \tag{1.2}$$

*Key words*: Cauchy problem, Klein-Gordon equation, Landweber method, potentials.

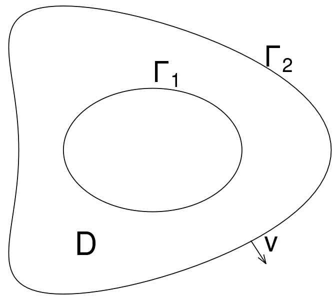



Fig. 1.1: Doubly connected domain

with $f_1$ and $f_2$ being sufficiently smooth functions.

Our objective is to reconstruct the Cauchy data on the inner boundary $\Gamma_1$, which is assumed to be inaccessible to direct measurements.

For a discussion of the uniqueness of the solution to the Cauchy problem defined by equation (1.1) and Cauchy data (1.2), the reader is referred to [5]. However, the solution generally does not exhibit continuous dependence on the input data. This lack of stability characterizes the problem as ill-posed in the sense of Hadamard, making the application of standard (classical) numerical methods unsuitable. In [8], the alternating method was applied to problem (1.1)–(1.2). In this work, we consider another iterative approach, the so-called Landweber method [2, 4, 6, 7] to solve the problem (1.1)–(1.2). Both the Landweber and alternating methods [3, 6] require solving two mixed boundary value problems per iteration. However, we apply the Landweber method, where both mixed problems are of the same type (Dirichlet–Neumann). This implies that the matrix of the linear system received by numerical solution of the integral equations associated with one problem is identical to that of the other, differing only in the right-hand side. Therefore, the matrix assembly needs to be performed only once and can be reused in all iterations. This reduces the computational cost compared to the alternating method, where two mixed problems require at least two distinct matrices to be assembled and stored. Additionally, our motivation is to employ the classical Landweber method for the problem (1.1)–(1.2) as a basis, with the possibility of further applying its modification [2], which in a special case will be equivalent to the alternating method.

The sections of this paper are organized as follows: Section 2 focuses on the Landweber method and its implementation. Section 3 presents the numerical solution of the mixed Dirichlet–Neumann problem using the boundary integral equations method. Numerical examples corresponding to well-posed mixed and ill-posed Cauchy problems for different domain configurations are discussed in Section 4. Finally, Section 5 concludes the paper with closing remarks.

## 2 A Landweber iterative method

The Landweber iteration is a classical iterative regularization method for solving ill-posed linear equations of the first kind [9]. Therefore, we reformulate the Cauchy problem to apply this approach.

We consider the following mixed boundary value problem

$$\Delta u - \kappa^2 u = 0 \quad \text{in } D, \tag{2.1}$$

$$u = h \quad \text{on } \Gamma_1, \qquad \frac{\partial u}{\partial \nu} = f_2 \quad \text{on } \Gamma_2. \tag{2.2}$$

Define a linear operator

$$Kh = u|_{\Gamma_2},$$

where $u$ is the solution to the mixed boundary value problem (2.1)–(2.2) with $f_2 = 0$.

Since the Cauchy problem admits a unique solution, the operator $K$ has a trivial kernel, i.e., $\ker(K) = \{0\}$. Further, without loss of generality, we assume that $f_2$ is zero. Therefore, the original problem (1.1)–(1.2) is equivalent to solving the operator equation

$$Kh = f_1. \tag{2.3}$$

We also introduce another mixed boundary value problem

$$\Delta v - \kappa^2 v = 0 \quad \text{in } D, \tag{2.4}$$

$$v = 0 \quad \text{on } \Gamma_1, \quad \frac{\partial v}{\partial \nu} = g \quad \text{on } \Gamma_2. \tag{2.5}$$

Using Green's identity, one can show that the adjoint of $K$ is given by

$$K^* g = -\left.\frac{\partial v}{\partial \nu}\right|_{\Gamma_1},$$

where $v$ solves the problem (2.4)–(2.5). We now present the steps that lead to this result. The adjoint operator $K^*$ is defined by the following relation

$$(Kh, g)_{\Gamma_2} = (h, K^* g)_{\Gamma_1}, \quad \forall h \in L^2(\Gamma_1),\ g \in L^2(\Gamma_2).$$

To derive the form of $K^*$, we use Green's second identity for two sufficiently smooth functions $u$ and $v$ defined on $D$

$$\int_D (u\Delta v - v\Delta u)\, dx = \int_{\partial D} \left( u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds,$$

where $u$ and $v$ solve the (2.1)–(2.2) with $f_2 = 0$ and (2.4)–(2.5), accordingly.

Substituting into Green's identity we obtain that

$$\int_D \left( u\kappa^2 v - v\kappa^2 u \right) dx = 0 \Rightarrow \int_{\partial D} \left( u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds = 0.$$

Taking into account boundary conditions we get

$$\int_{\Gamma_1} h\frac{\partial v}{\partial \nu}\, ds + \int_{\Gamma_2} ug\, ds = 0,$$

which is equal to

$$(Kh, g)_{\Gamma_2} = (h, -\tfrac{\partial v}{\partial \nu})_{\Gamma_1}.$$

By the definition of the adjoint operator, the above identity shows that

$$K^* g = -\left.\frac{\partial v}{\partial \nu}\right|_{\Gamma_1}.$$

The classical Landweber method for solving the operator equation (2.3) is defined by the following iterative scheme

$$h_k = h_{k-1} - \eta K^*(Kh_{k-1} - f_1),$$

where $\eta > 0$ – is a relaxation parameter and $K^*$ denotes the adjoint operator of $K$.

Thus, the Landweber method applied to the Cauchy problem (1.1)–(1.2) consists of iteratively solving two mixed boundary value problems on the domain $D$: one to evaluate $Kh$, and another to compute $K^* g$. Let $f_1$ and $f_2$ be as defined in the original Cauchy problem. The iterative procedure for approximating the solution to the problem (1.1)-(1.2) is as follows:

- Choose an arbitrary initial guess $h_0$. Solve the first mixed boundary value problem (2.1)–(2.2) to obtain $u_0$, using the condition $u_0 = h_0$ on $\Gamma_1$.
- Solve the problem (2.4)–(2.5) to find $v_0$, where the Neumann condition on $\Gamma_2$ is given by

$$\frac{\partial v_0}{\partial \nu} = g_0$$

with $g_0 = u_0 - f_1$.
- For iteration $k \geq 1$, assume $u_{k-1}$ and $v_{k-1}$ are known. Then
  - Define
$$h_k = h_{k-1} - \eta \left.\frac{\partial v_{k-1}}{\partial \nu}\right|_{\Gamma_1}, \quad \eta > 0.$$
  - Solve the problem (2.1)–(2.2) with $u_k = h_k$ on $\Gamma_1$ to obtain $u_k$.
  - Set
$$g_k = u_k - f_1$$
and solve the problem (2.4)–(2.5) for $v_k$, using $\frac{\partial v_k}{\partial \nu} = g_k$ on $\Gamma_2$.

The iterative process continues by repeating the final step described above. As a stopping criterion, the discrepancy principle can be applied to ensure stability in the presence of noise.

Note, that while the operator $K$ is defined using a homogeneous Neumann data on $\Gamma_2$, the actual problem (2.1)–(2.2) in the Landweber iteration is solved using the original (non-zero) boundary condition $f_2$. The presence of non-zero $f_2$ can be incorporated by decomposing the solution into a particular part corresponding to $f_2$ and a homogeneous part. Therefore, solving the problem (2.1)–(2.2) with $f_2 \neq 0$ in each iteration is consistent with the original Cauchy problem and leads to convergence of the method.

The convergence result follows from the general theory of the Landweber iteration for solving ill-posed linear operator equations [9].

**Theorem 2.1.** *Assume that problem (1.1)-(1.2) has a bounded solution with trace $h$ on the boundary $\Gamma_1$. Let $h_k$ be the $k$-th approximate solution on the boundary $\Gamma_1$ given by the procedure described above. Then, for sufficiently small $\eta > 0$ and for any smooth initial data $h_0$,*

$$\lim_{k\to\infty} \|h - h_k\|_{L^2(\Gamma_1)} = 0.$$

It is worth mentioning that the magnitude of $\eta$ can be estimated based on the norm of the operator $K$ [9]. Also, note that convergence of the Landweber method in other weight spaces is considered in [10].

## 3 An integral equation method for the Dirichlet-Neumann problem

The mixed boundary value problem (2.1)–(2.2), or similarly (2.4)–(2.5), will be addressed by reducing it to a system of second-kind boundary integral equations. The solution is represented as a combination of single-layer and double-layer potentials

$$u(x) = \int_{\Gamma_1} \mu_1(y) \frac{\partial \Phi(\kappa; x, y)}{\partial \nu(y)} ds(y) + \int_{\Gamma_2} \mu_2(y) \Phi(\kappa; x, y) ds(y), \quad x \in D \tag{3.1}$$

with unknown densities $\mu_1 \in C(\Gamma_1)$ and $\mu_2 \in C(\Gamma_2)$. Here $\Phi(\kappa; x, y) = \frac{1}{2\pi} K_0(\kappa|x-y|)$ – fundamental solution of the operator (1.1), where $K_0$ is the modified Bessel function of the second kind and zeroth order [1]

$$K_0(z) = \ln \frac{1}{z} I_0(z) + \sum_{k=0}^{\infty} (\ln 2 + \psi(k+1)) \frac{(z^2/4)^k}{(k!)^2},$$

where

$$I_0(z)=\sum_{k=0}^{\infty}\frac{(z^2/4)^k}{(k!)^2},\quad \psi(1)=-\gamma,\quad \psi(n)=-\gamma+\sum_{k=1}^{n-1}k^{-1},\ n\geq 2.$$

Here $\gamma$ is the Euler's constant. Note that $I_0'(z)=I_1(z),\ K_0'(z)=-K_1(z)$, where $K_1(z)$ – modified Bessel function of the 2-nd kind and 1-st order and $I_0(z),I_1(z)$ are the modified Bessel function of the 1-st kind of 0-th and 1-st order, respectively [1].

By matching the representation (3.1) with the given boundary conditions (2.2), and using well-known properties of single and double-layer potentials and their derivatives on the boundary of the domain $D$ we obtain the following system of integral equations of the second kind

$$\begin{cases} \left(\frac{1}{2}I+\mathcal{D}_{11}\right)\mu_1+\mathcal{S}_{21}\mu_2=h & \text{on } \Gamma_1,\\ \mathcal{T}_{12}\mu_1+\left(\frac{1}{2}I+\mathcal{Q}_{22}\right)\mu_2=f_2 & \text{on } \Gamma_2 \end{cases} \tag{3.2}$$

with introduced operators

$$(\mathcal{S}_{kp}\varphi)(x)=\int_{\Gamma_k}\varphi(y)\Phi(\kappa;x,y)ds(y),\quad x\in\Gamma_p,$$

$$(\mathcal{D}_{kp}\varphi)(x)=\int_{\Gamma_k}\varphi(y)\frac{\partial\Phi(\kappa;x,y)}{\partial\nu(y)}ds(y),\quad x\in\Gamma_p,$$

$$(\mathcal{T}_{kp}\varphi)(x)=\frac{\partial}{\partial\nu(x)}\int_{\Gamma_k}\varphi(y)\frac{\partial\Phi(\kappa;x,y)}{\partial\nu(y)}ds(y),\quad x\in\Gamma_p,$$

$$(\mathcal{Q}_{kp}\varphi)(x)=\int_{\Gamma_k}\varphi(y)\frac{\partial\Phi(\kappa;x,y)}{\partial\nu(x)}ds(y),\quad x\in\Gamma_p$$

for $k,p=1,2$.

Note, that in case of $k=p$ using Maue transformation, similarly to what was used in [11], $(\mathcal{T}_{kk}\varphi)(x)$ can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial\nu(x)}\int_{\Gamma_k}\varphi(y)\frac{\partial\Phi(\kappa;x,y)}{\partial\nu(y)}ds(y)&=\int_{\Gamma_k}\frac{\partial\varphi}{\partial\theta}(y)\frac{\partial\Phi(\kappa;x,y)}{\partial\theta(x)}ds(y)-\\ &-\kappa^2\int_{\Gamma_k}\varphi(y)\Phi(\kappa;x,y)[\nu(x)\cdot\nu(y)]ds(y),\quad x\in\Gamma_k,\end{aligned} \tag{3.3}$$

where $\theta$ - denotes unit tangential vector, counterclockwise oriented.

**Theorem 3.1.** *Let $\Gamma_1$ and $\Gamma_2\in C^2$. Assume the boundary data satisfy $h\in C(\Gamma_1),\ f_2\in C(\Gamma_2)$. Then the system of boundary integral equations* (3.2) *of the second kind admits a unique solution $\mu_k\in C(\Gamma_k),\ k=1,2$, which depends continuously on the given data.*

*Proof.* Since operators in (3.2) contain continuous and weakly singular kernels, therefore they are compact in $C(\Gamma_1)$ and $C(\Gamma_2)$. Then from the Riesz theory [12] and the uniqueness of the solution of the problem (2.1)–(2.2) there exists a unique solution that depends continuously on the right side. □

**Remark 3.2.** *For the Landweber iteration, one needs to evaluate the normal derivative of the solution represented in* (3.1) *on the interior boundary $\Gamma_1$, which involves hypersingular integral* (3.3) *For the numerical computation we use the right side of* (3.3), *so the density $\mu_1$ must possess at least one continuous derivative. Therefore, in practice, we assume $\mu_1\in C^1(\Gamma_1)$, and in this case we require $h\in C^1(\Gamma_1)$, and the boundary $\Gamma_1\in C^3$.*

Let

$$\Gamma_k=\{x_k(t)=(x_{k1}(t),x_{k2}(t)),\ t\in[0,2\pi]\},\quad k=1,2,$$

where $x_k : \mathbb{R} \to \mathbb{R}^2$, $x_1 \in C^3[0,2\pi] \times C^3[0,2\pi]$, $x_2 \in C^2[0,2\pi] \times C^2[0,2\pi]$ and $2\pi$-periodic with $|x_k'(t)| > 0$ for all $t \in [0,2\pi]$.

Taking into account the parametric representation of $\Gamma_k$ we rewrite the system (3.2) in the following equivalent parametric form

$$\begin{cases} \Big(\dfrac{1}{2}I + \widetilde{\mathcal{D}}_{11}\Big)\phi_1 + \widetilde{\mathcal{S}}_{21}\phi_2 = \widetilde{h}, \\ \widetilde{\mathcal{T}}_{12}\phi_1 + \Big(\dfrac{1}{2}I + \widetilde{\mathcal{Q}}_{22}\Big)\phi_2 = \widetilde{f}_2 \end{cases} \tag{3.4}$$

with parametrised integral operators

$$(\widetilde{\mathcal{S}}_{kp}\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau) S_{kp}(t,\tau)d\tau, \quad (\widetilde{\mathcal{D}}_{kp}\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau) D_{kp}(t,\tau)d\tau,$$

$$(\widetilde{\mathcal{Q}}_{kp}\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau) Q_{kp}(t,\tau)d\tau, \quad (\widetilde{\mathcal{T}}_{kp}\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau) T_{kp}(t,\tau)d\tau,$$

where $\phi_k(t) = \mu_k(x_k(t))$, $k,p = 1,2$; $\quad \widetilde{h}(t) = h(x_1(t))$, $\widetilde{f}_2(t) = f_2(x_2(t))$, $t \in [0,2\pi]$.

Considering functions $K_0$ and $K_1$, the kernels of the integrals operators of the system (3.4), including those in which the logarithmic singularity is concentrated, can be presented in the following form

$$S_{kp}(t,\tau) = K_0(\kappa|x_p(t) - x_k(\tau)|)|x_k'(\tau)|, \quad p \neq k,$$

$$S_{kk}(t,\tau) = S_{kk}^1(t,\tau)\ln\Big(4\sin^2\frac{t-\tau}{2}\Big) + S_{kk}^2(t,\tau)$$

with

$$S_{kk}^1(t,\tau) = -\frac{1}{2}I_0(\kappa|x_k(t) - x_k(\tau)|)|x_k'(\tau)|,$$

$$S_{kk}^2(t,\tau) = \begin{cases} S_{kk}(t,\tau) - S_{kk}^1(t,\tau)\ln\Big(4\sin^2\dfrac{t-\tau}{2}\Big), & t \neq \tau, \\ -\dfrac{1}{2}\ln\Big(\dfrac{\kappa^2|x_k'(t)|^2}{4}\Big)|x_k'(t)| - \gamma|x_k'(t)|, & t = \tau; \end{cases}$$

$$D_{kp}(t,\tau) = \kappa K_1(\kappa|x_p(t) - x_k(\tau)|)\frac{(x_p(t) - x_k(\tau)) \cdot \nu(x_k(\tau))}{|x_p(t) - x_k(\tau)|}|x_k'(\tau)|, \quad p \neq k,$$

$$D_{kk}(t,\tau) = D_{kk}^1(t,\tau)\ln\Big(4\sin^2\frac{t-\tau}{2}\Big) + D_{kk}^2(t,\tau)$$

with

$$D_{kk}^1(t,\tau) = \begin{cases} \dfrac{1}{2}I_1(\kappa|x_k(t) - x_k(\tau)|)\dfrac{(x_k(t) - x_k(\tau)) \cdot \nu(x_k(\tau))}{|x_k(t) - x_k(\tau)|}|x_k'(\tau)|, & t \neq \tau, \\ 0, & t = \tau, \end{cases}$$

$$D_{kk}^2(t,\tau) = \begin{cases} D_{kk}(t,\tau) - D_{kk}^1(t,\tau)\ln\Big(4\sin^2\dfrac{t-\tau}{2}\Big) & t \neq \tau, \\ \dfrac{x_k''(t) \cdot \nu(x_k(t))}{2|x_k'(t)|}, & t = \tau; \end{cases}$$

$$Q_{kp}(t,\tau) = -\kappa K_1(\kappa|x_p(t) - x_k(\tau)|)\frac{(x_p(t) - x_k(\tau)) \cdot \nu(x_k(t))}{|x_p(t) - x_k(\tau)|}|x_k'(\tau)|, \quad p \neq k,$$

$$Q_{kk}(t,\tau) = Q^1_{kk}(t,\tau)\ln\Big(4\sin^2\frac{t-\tau}{2}\Big) + Q^2_{kk}(t,\tau)$$

with

$$Q^1_{kk}(t,\tau) = \begin{cases} -\dfrac{1}{2}I_1(\kappa|x_k(t)-x_k(\tau)|)\dfrac{(x_k(t)-x_k(\tau))\cdot\nu(x_k(t))}{|x_k(t)-x_k(\tau)|}|x'_k(\tau)|, & t\neq\tau,\\ 0, & t=\tau,\end{cases}$$

$$Q^2_{kk}(t,\tau) = \begin{cases} Q_{kk}(t,\tau) - Q^1_{kk}(t,\tau)\ln\Big(4\sin^2\dfrac{t-\tau}{2}\Big) & t\neq\tau,\\ \dfrac{x''_k(t)\cdot\nu(x_k(t))}{2|x'_k(t)|}, & t=\tau;\end{cases}$$

$$T_{kp}(t,\tau) = |x'_k(\tau)|\Bigg(\kappa^2 N_1(t,\tau)\Big[K_0(\kappa|x_p(t)-x_k(\tau)|) + \frac{K_1(\kappa|x_p(t)-x_k(\tau)|)}{\kappa|x_p(t)-x_k(\tau)|}\Big] - \\ -\kappa N_2(t,\tau)K_1(\kappa|x_p(t)-x_k(\tau)|)\Bigg), \quad p\neq k,$$

where

$$N_1(t,\tau) = -\frac{[(x_p(t)-x_k(\tau))\cdot\nu(x_k(t))][(x_p(t)-x_k(\tau))\cdot\nu(x_p(\tau))]}{|x_p(t)-x_k(\tau)|^2},$$

$$N_2(t,\tau) = -\frac{\nu(x_p(t))\cdot\nu(x_k(\tau))}{|x_p(t)-x_k(\tau)|} - \frac{N_1(t,\tau)}{|x_p(t)-x_k(\tau)|}.$$

Considering the identity (3.3), the integral operator $\widetilde{\mathcal{T}}_{kk}$ can be rewritten in the form provided below

$$(\widetilde{\mathcal{T}}_{kk}\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{1}{2|x'_k(t)|}\phi'(\tau)\cot\frac{\tau-t}{2} + \phi(\tau)T_{kk}(t,\tau)\Big]d\tau$$

with

$$T_{kk}(t,\tau) = H^1_{kk}(t,\tau)\ln\Big(4\sin^2\frac{t-\tau}{2}\Big) + H^2_{kk}(t,\tau),$$

where

$$H^1_{kk}(t,\tau) = -\widetilde{L}^{(1)}_{k1}(t,\tau) - L^{(1)}_{k2}(t,\tau), \quad H^2_{kk}(t,\tau) = -\widetilde{L}^{(2)}_{k1}(t,\tau) - L^{(2)}_{k2}(t,\tau),$$

having

$$\widetilde{L}^{(1)}_{k1}(t,\tau) = \frac{\kappa^2}{2}I_0(\kappa|x_k(t)-x_k(\tau)|)q_1(t,\tau) + \\ \frac{\kappa}{2}I_1(\kappa|x_k(t)-x_k(\tau)|)\Big[\frac{x'_k(\tau)\cdot\theta(x_k(t))}{|x_k(t)-x_k(\tau)|} - \frac{2q_1(t,\tau)}{|x_k(t)-x_k(\tau)|}\Big], \quad t\neq\tau,$$

$$\widetilde{L}^{(1)}_{k1}(t,t) = \frac{\kappa^2}{4}|x'_k(t)|,$$

$$\widetilde{L}^{(2)}_{k1}(t,\tau) = \widetilde{L}_{k1}(t,\tau) - \widetilde{L}^{(1)}_{k1}(t,\tau)\ln\Big(4\sin^2\frac{t-\tau}{2}\Big), \quad t\neq\tau,$$

$$\widetilde{L}^{(2)}_{k1}(t,t) = \frac{1}{4}\kappa^2|x'_k(t)| + \frac{1}{2}\kappa^2|x'_k(t)|\Big\{\frac{1}{2}\ln\frac{\kappa^2|x'_k(t)|^2}{4} + \gamma\Big\} + \\ \frac{1}{|x'_k(t)|}\Big\{\frac{1}{12} - \frac{1}{6}\frac{x'_k(t)\cdot x'''_k(t)}{|x'_k(t)|^2} - \frac{1}{4}\frac{x''_k(t)\cdot x''_k(t)}{|x'_k(t)|^2} + \frac{(x'_k(t)\cdot x''_k(t))^2}{2|x'_k(t)|^4}\Big\}$$

denoting

$$q_1(t,\tau) = \frac{(x_k(t)-x_k(\tau))\cdot\theta(x_k(t))(x_k(t)-x_k(\tau))\cdot x_k'(\tau)}{|x_k(t)-x_k(\tau)|^2},\ t\neq\tau,\quad q_1(t,t)=|x_k'(t)|.$$

Also

$$\widetilde{L}_{k1}(t,\tau) = K_0(\kappa|x_k(t)-x_k(\tau)|)\Big[-\kappa^2\frac{(x_k(t)-x_k(\tau))\cdot\theta(x_k(t))(x_k(t)-x_k(\tau))\cdot x_k'(\tau)}{|x_k(t)-x_k(\tau)|^2}\Big]+$$
$$K_1(\kappa|x_k(t)-x_k(\tau)|)\Big[\kappa\frac{x_k'(\tau)\cdot\theta(x_k(t))}{|x_k(t)-x_k(\tau)|}-2\kappa\frac{(x_k(t)-x_k(\tau))\cdot\theta(x_k(t))(x_k(t)-x_k(\tau))\cdot x_k'(\tau)}{|x_k(t)-x_k(\tau)|^3}\Big]+$$
$$\frac{1}{4|x_k'(t)|\sin^2\frac{t-\tau}{2}},$$

$$L_{k2}^{(1)}(t,\tau) = -\frac{\kappa^2}{2}I_0(\kappa|x_k(t)-x_k(\tau)|)q_2(t,\tau),\quad L_{k2}^{(1)}(t,t) = -\frac{\kappa^2}{2}|x_k'(t)|,$$

$$L_{k2}^{(2)}(t,\tau) = L_{k2}(t,\tau) - L_{k2}^{(1)}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right),\quad t\neq\tau$$

with

$$L_{k2}^{(2)}(t,t) = \kappa^2|x_k'(t)|\Big(-\frac{1}{2}\ln\Big(\frac{\kappa^2|x_k'(t)|^2}{4}\Big)-\gamma\Big),$$

$$L_{k2}(t,\tau) = \kappa^2 K_0(\kappa|x-y|)[\nu(x_k(t))\cdot\nu(x_k(\tau))]|x_k'(\tau)|,$$

$$q_2(t,\tau) = [\nu(x_k(t))\cdot\nu(x_k(\tau))]|x_k'(\tau)|.$$

Hence, the explicit view of (3.4) for $t\in[0,2\pi]$ will be

$$\begin{cases} \frac{1}{2}\phi_1(t)+\frac{1}{2\pi}\int\limits_0^{2\pi}\Big[\phi_1(\tau)\Big(D_{11}^1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)+D_{11}^2(t,\tau)\Big)+\phi_2(\tau)S_{21}(t,\tau)\Big]d\tau = \widetilde{h}(t),\\ \frac{1}{2}\phi_2(t)+\frac{1}{2\pi}\int\limits_0^{2\pi}\Big[\phi_1(\tau)T_{12}(t,\tau)+\phi_2(\tau)\Big(Q_{22}^1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)+Q_{22}^2(t,\tau)\Big)\Big]d\tau = \widetilde{f}_2(t). \end{cases} \tag{3.5}$$

Based on the well-posedness of (3.2), the system (3.5) has a unique solution that belongs to the space of $2\pi$-periodic continuous functions and continuously depends on the right-hand side.

Full discretisation of the system (3.5) can be obtained using the Nyström method [12]. To this end we split the interval $[0,2\pi]$ with the nodes

$$t_j = \frac{j\pi}{n},\quad j=0,...,2n-1,\ n\in\mathbb{N},$$

apply the quadratures, see [8, 12, 13]

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau \approx \frac{1}{2n}\sum_{j=0}^{2n-1} f(t_j),$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)d\tau \approx \sum_{j=0}^{2n-1} f(t_j)R_j(t),$$

$$\frac{1}{2\pi}\int_0^{2\pi} f'(\tau)\cot\frac{\tau-t}{2}d\tau \approx \sum_{j=0}^{2n-1} f(t_j)T_j(t)$$

with weight functions

$$R_j(t) = -\frac{1}{n}\left\{\sum_{i=1}^{n-1}\frac{1}{i}\cos i\,(t-t_j) + \frac{1}{2n}\cos n\,(t-t_j)\right\},$$

$$T_j(t) = -\frac{1}{n}\sum_{i=1}^{n-1} i\cos i\,(t-t_j) - \frac{1}{2}\cos n\,(t-t_j),$$

$j = 0,..,2n-1$, and collocating in nodes $t_j$

$$\begin{cases} \dfrac{1}{2}\tilde{\phi}_{1i} + \dfrac{1}{2n}\displaystyle\sum_{i=0}^{2n-1}\Big[\tilde{\phi}_{1j}\Big(D^1_{11}(t_i,t_j)R_j(t_i) + D^2_{11}(t_i,t_j)\Big) + \tilde{\phi}_{2j}S_{21}(t_i,t_j)\Big] = \widetilde{h}_i, \\ \dfrac{1}{2}\tilde{\phi}_{2i} + \dfrac{1}{2n}\displaystyle\sum_{i=0}^{2n-1}\Big[\tilde{\phi}_{1j}T_{12}(t_i,t_j) + \tilde{\phi}_{2j}\Big(Q^1_{22}(t_i,t_j)R_j(t_i) + Q^2_{22}(t_i,t_j)\Big)\Big] = \widetilde{f}_{2i}, \end{cases}$$

where $\tilde{\phi}_{ki} \approx \phi_k(t_i), k = 1,2$ are unknown approximated values of the densities, $\widetilde{h}_i = \widetilde{h}(t_i)$, $\widetilde{f}_{2i} = \widetilde{f}_2(t_i)$, $i = 0,\dots,2n-1$.

The approximate solution $\tilde{u}$ at any point in the domain can be obtained from the following formula, using the approximate values of the densities

$$u(x) \approx \tilde{u}(x) = \frac{1}{2n}\sum_{j=0}^{2n-1}\Big[\tilde{\phi}_{1j}\Lambda(\kappa;x,t_j) + \tilde{\phi}_{2j}\tilde{\Lambda}(\kappa;x,t_j)\Big], \quad x \in D,$$

where

$$\Lambda(\kappa;x,t_j) = \kappa K_1(\kappa|x - x_1(t_j)|)\frac{(x - x_1(t_j))\cdot\nu(x_1(t_j))}{|x - x_1(t_j)|}|x_1'(t_j)|,$$

$$\tilde{\Lambda}(\kappa;x,t_j) = K_0(\kappa|x - x_2(t_j)|)|x_2'(t_j)|.$$

By taking the trace on $\Gamma_2$ of the solution given by (3.1) and its normal derivative, and employing the jump relations of the potentials, we obtain

$$u(x) = \int_{\Gamma_1}\mu_1(y)\frac{\partial\Phi(\kappa;x,y)}{\partial\nu(y)}ds(y) + \int_{\Gamma_2}\mu_2(y)\Phi(\kappa;x,y)ds(y), \quad x \in \Gamma_2,$$

$$\frac{\partial u}{\partial\nu}(x) = \frac{\partial}{\partial\nu(x)}\int_{\Gamma_1}\mu_1(y)\frac{\partial\Phi(\kappa;x,y)}{\partial\nu(y)}ds(y) + \int_{\Gamma_2}\mu_2(y)\frac{\partial\Phi(\kappa;x,y)}{\partial\nu(x)}ds(y), \quad x \in \Gamma_1.$$

The parametric representations will be

$$u(x_2(t)) = \frac{1}{2\pi}\int_0^{2\pi}\Big[\phi_1(\tau)D_{12}(t,\tau) + \phi_2(\tau)S_{22}(t,\tau)\Big]d\tau,$$

$$\frac{\partial u}{\partial\nu}(x_1(t)) = \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{1}{2|x_1'(t)|}\phi_1'(\tau)\cot\frac{\tau-t}{2} +$$
$$+\phi_1(\tau)\Big(H^1_{11}(t,\tau)\ln\Big(4\sin^2\frac{t-\tau}{2}\Big) + H^2_{11}(t,\tau)\Big) + \phi_2(\tau)Q_{21}(t,\tau)\Big]d\tau.$$

The corresponding discrete approximations can be obtained using the quadratures mentioned above

$$u(x_2(t)) \approx \tilde{u}(x_2(t)) = \frac{1}{2n}\sum_{j=0}^{2n-1}\left[\tilde{\phi}_{1j}D_{12}(t,t_j) + \tilde{\phi}_{2j}S_{22}(t,t_j)\right]$$

and

$$\frac{\partial u}{\partial \nu}(x_1(t)) \approx \frac{\partial \tilde{u}}{\partial \nu}(x_1(t)) = \sum_{j=0}^{2n-1}\left[\frac{1}{2|x_1'(t)|}\tilde{\phi}_{1j}T_j(t)+\right.$$
$$\left.+\tilde{\phi}_{1j}\left(H^1_{11}(t,t_j)R_j(t) + \frac{1}{2n}H^2_{11}(t,t_j)\right) + \frac{1}{2n}\tilde{\phi}_{2j}Q_{21}(t,t_j)\right].$$

The convergence analysis and error estimates for the applied numerical method can be justified using the theory of collectively compact operators and the approximation properties of trigonometric interpolation in appropriate Banach spaces (see [12]). Note that exponential convergence of the approximation error is expected when both the boundary and the boundary data are analytic.

## 4 Numerical examples

**Example 4.1.** Let's define the domain $D$ using the following curves (see Fig. 4.1)

$$\Gamma_1 = \{x_1(t) = (0.5\cos(t), 0.4\sin(t) - 0.3\sin^2(t)),\ t \in [0, 2\pi]\}$$

and

$$\Gamma_2 = \{x_2(t) = (0.9\cos(t), 0.4 + 0.9\sin(t) - 0.75\sin^2(t)),\ t \in [0, 2\pi]\}.$$

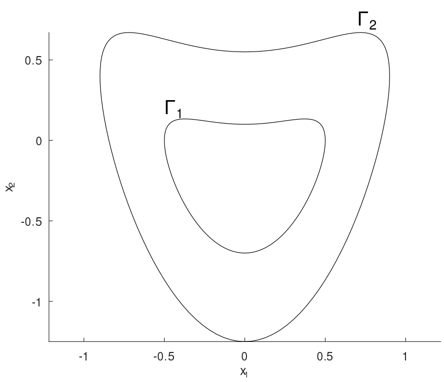



Fig. 4.1: Domain $D$ for the example 4.1

Considering the mixed Dirichlet-Neumann problem (2.1)–(2.2) we take that $u_{ex}(x) = \Phi(x, y^*)$, with $y^* = (1,1)$ and $\kappa = 1$, so the boundary conditions are: $h$ is a trace of $u_{ex}$ on $\Gamma_1$ and $f_2$ is defined as the normal derivative of the $\Phi(x, y^*)$ on $\Gamma_2$ or in other words

$$f_2(x) = \frac{-\kappa(x - y^*)\cdot \nu(x)K_1(\kappa|x - y^*|))}{2\pi|x - y^*|}, \quad x \in \Gamma_2.$$

Table 4.1 presents the maximum norm errors of the computed normal derivative values on $\Gamma_1$ and calculated numerical solution on $\Gamma_2$ for various discretisation parameters $n$. Additionally, the absolute error of the solution is provided at a point $x^* = (0.5, 0.4)$.

*Table 4.1.* Errors for the mixed Dirichlet-Neumann problem in the Ex. 4.1

| $n$ | $\lvert u_{ex}(x^*) - \tilde{u}(x^*)\rvert$ | $\lVert \frac{\partial u_{ex}}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu} \rVert_{C(\Gamma_1)}$ | $\lVert u_{ex} - \tilde{u} \rVert_{C(\Gamma_2)}$ |
|---|---|---|---|
| 8 | 4.99E-05 | 1.62E-03 | 1.79E-03 |
| 16 | 6.25E-07 | 3.53E-05 | 1.58E-05 |
| 32 | 6.46E-11 | 4.01E-08 | 4.42E-09 |
| 64 | 2.78E-17 | 5.59E-14 | 4.72E-16 |

**Example 4.2.** To generate the input Cauchy data for the problem (1.1)-(1.2), we first solve the mixed Dirichlet-Neumann problem using predefined input functions. This allows us to compute the corresponding Cauchy data on both boundaries. As a result, we obtain the necessary input data for (1.1)-(1.2), along with the solution and its normal derivative values on the inner boundary. These values serve as reference data for validating the accuracy of the approximate solution. Note that the constant $\kappa$ equals 1 for this and the next example.

The domain $D$ is defined by two curves (see Fig 4.2)

$$\Gamma_1 = \{x_1(t) = (0.6\cos(t) + 0.2\cos(2t), 0.6\sin(t),\ t \in [0, 2\pi]\},$$

$$\Gamma_2 = \{x_2(t) = (1.2\cos(t), 1.2\sin(t),\ t \in [0, 2\pi]\}.$$

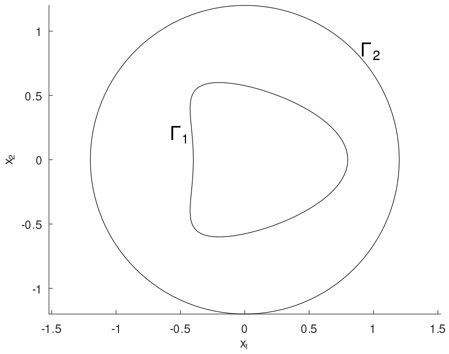



Fig. 4.2: Domain $D$ for the example 4.2

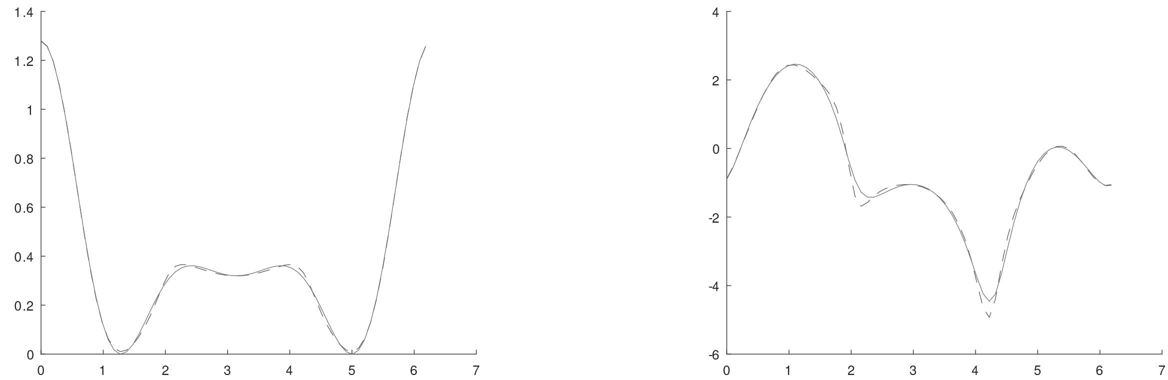
In order to initialize the Cauchy data we solve (2.1)–(2.2) with $h(x) = 2x_1^2,\ x \in \Gamma_1$ and $f_2(x) = x_1 + x_2,\ x \in \Gamma_2$ and $n = 128$. As for the Landweber method we take initial guess as zero and the parameter $\eta = 0.3$ with $n = 32$ for the mixed problems. Note, that input data with noise were generated by the following formula

$$f_2^\delta = f_2 + \delta(2\beta - 1)\|f_2\|_{L^2(\Gamma_2)},$$

where $\delta$ is a noisy level (for instance, 4%), $\beta \in (0, 1)$ is some random number. The norm in $L^2$ can be calculated approximately

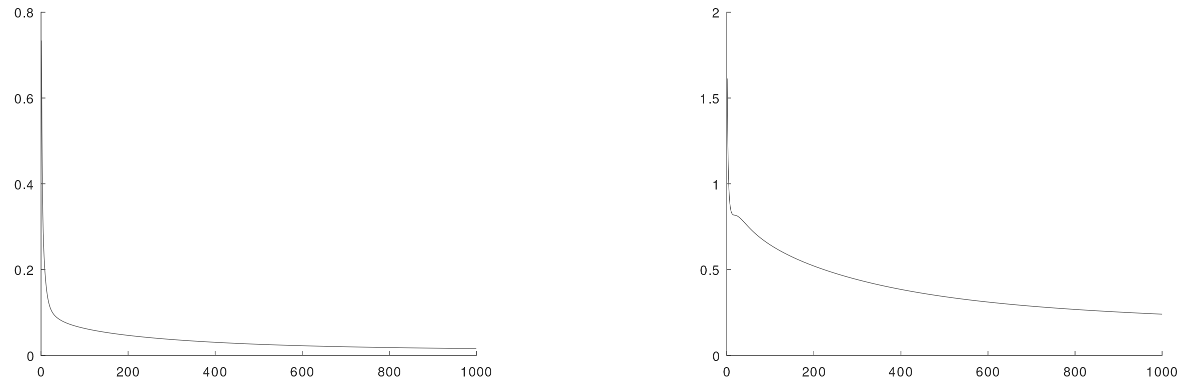
$$||f_2||_{L^2(\Gamma_2)} \approx \sqrt{\frac{\pi}{n} \sum_{j=0}^{2n-1} f_2(x_2(t_j))^2 |x_2'(t_j)|}.$$

The results of the iterative approach in Fig. 4.3, Fig. 4.5 illustrate the reconstruction of the function and its normal derivative for both exact input data and input data with noise. The solid line



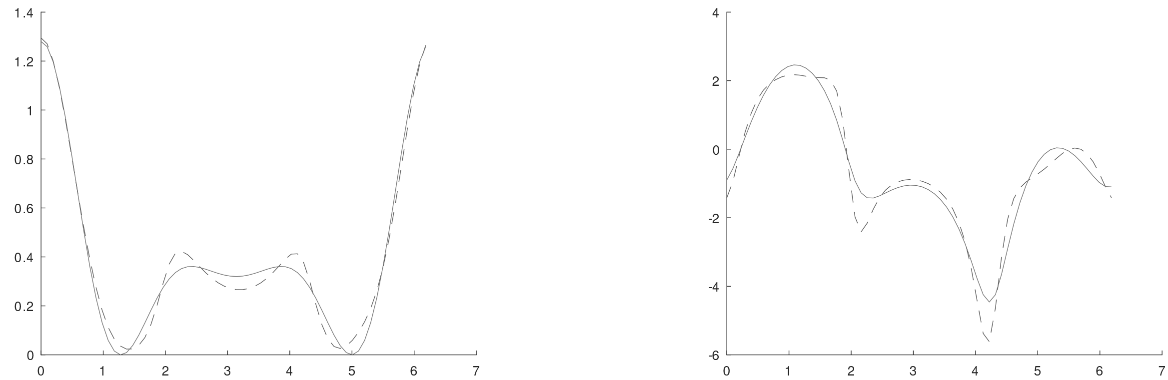

(a) boundary function on $\Gamma_1$ (b) normal derivative on $\Gamma_1$

Fig. 4.3: The exact (–) and approximate (- - -) solutions for the Ex. 4.2; exact input data, $k^* = 1000$. $\|u_{ex} - \tilde{u}\|_{L^2(\Gamma_1)} \approx 0.016$, $\|\frac{\partial u_{ex}}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu}\|_{L^2(\Gamma_1)} \approx 0.241$.



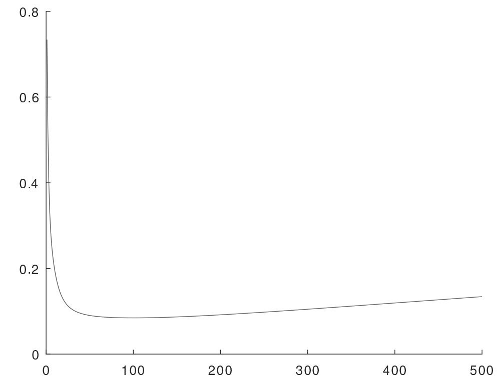

(a) $L^2$-error for the function on $\Gamma_1$ (b) $L^2$-error for the normal derivative on $\Gamma_1$

Fig. 4.4: $L^2$-errors for 1000 iterations for the Ex. 4.2; exact input data

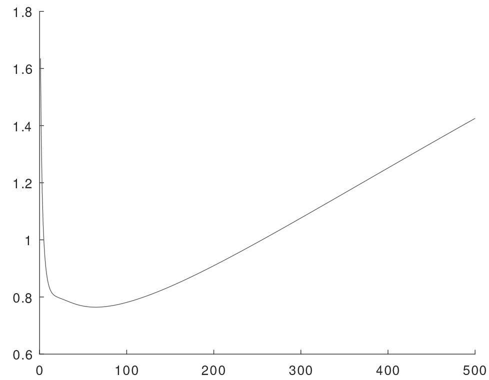



(a) boundary function on $\Gamma_1$, $k^* = 100$ (b) normal derivative on $\Gamma_1$, $k^* = 64$

Fig. 4.5: The exact (–) and approximate (- - -) solutions for the Ex. 4.2; noisy input data (4%). $\|u_{ex} - \tilde{u}\|_{L^2(\Gamma_1)} \approx 0.103$, $\|\frac{\partial u_{ex}}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu}\|_{L^2(\Gamma_1)} \approx 0.782$.



(a) $L^2$-error for the function on $\Gamma_1$

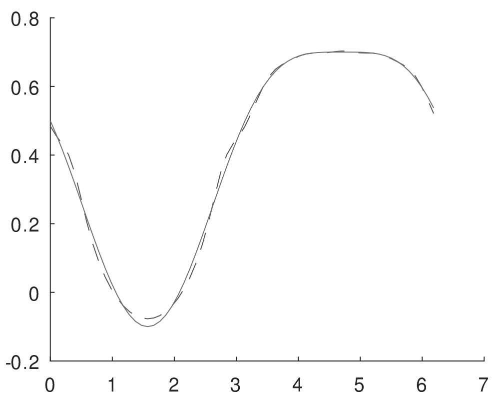


(b) $L^2$-error for the normal derivative on $\Gamma_1$

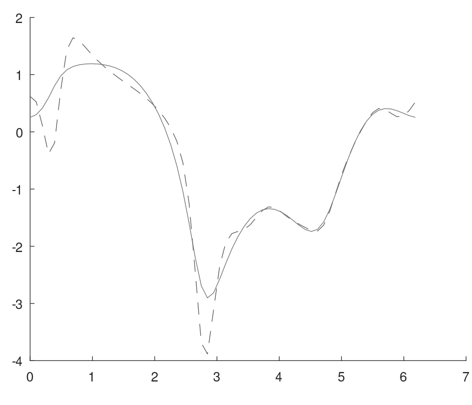
Fig. 4.6: $L^2$-errors for 500 iterations for the Ex. 4.2; noisy input data

(—) represents the graph of the exact solution, while the dashed line (- - -) indicates the approximate numerical solution. Also, $k^*$ denotes a number of the iteration with the best solution.

**Example 4.3.** Let's take $\Gamma_1$ and $\Gamma_2$ the same as from the example 4.1. The input functions for this numerical example are obtained by solving the mixed boundary value problem (2.1)–(2.2) with $h(x) = 2x_1^2 - x_2$, $x \in \Gamma_1$ and $f_2(x) = \sin(3x_1 + x_2)$, $x \in \Gamma_2$. The values $n$ (to generate the Cauchy data and to solve mixed problems), $\eta$ and initial guess $h_0$ were taken the same as in the previous example. Fig. 4.7 and Fig. 4.9 illustrate the reconstruction of the Cauchy data on $\Gamma_1$ in case of exact and noisy input data, while Fig. 4.8 and Fig. 4.10 show the associated $L_2$-errors at each iteration step.

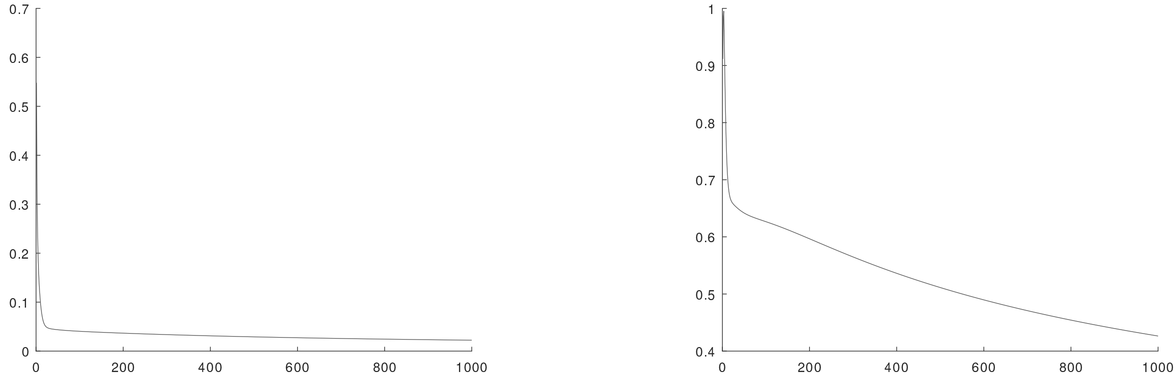


(a) boundary function on $\Gamma_1$



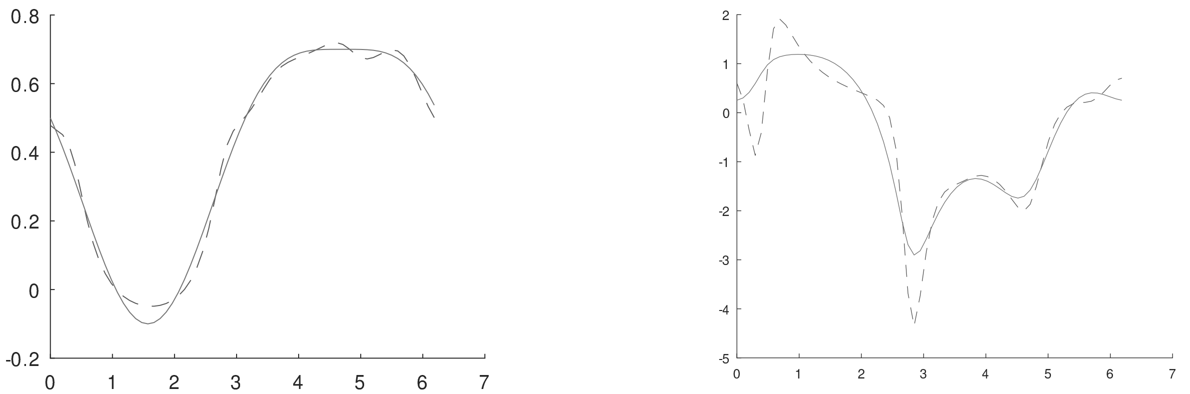
(b) normal derivative on $\Gamma_1$

Fig. 4.7: The exact (–) and approximate (- - -) solutions for the Ex. 4.3; exact input data, $k^* = 1000$. $\|u_{ex} - \tilde{u}\|_{L^2(\Gamma_1)} \approx 0.022$, $\|\frac{\partial u_{ex}}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu}\|_{L^2(\Gamma_1)} \approx 0.426$



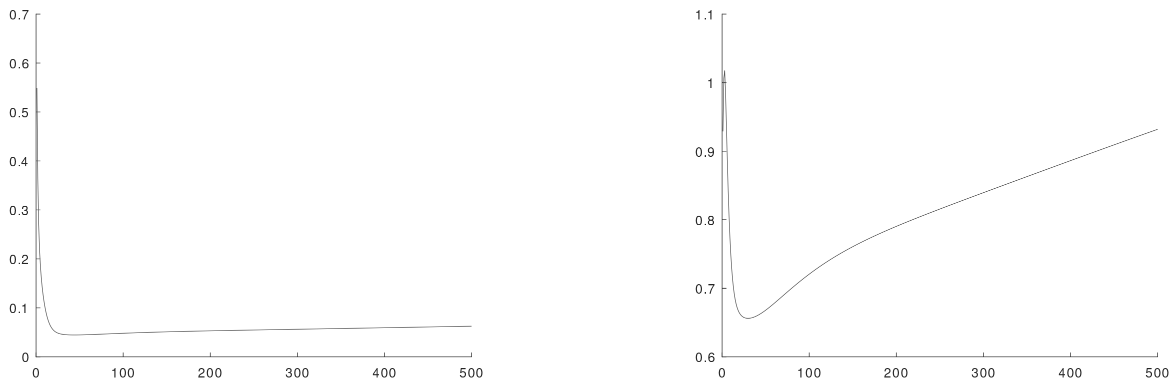

(a) $L^2$-error for the function on $\Gamma_1$ (b) $L^2$-error for the normal derivative on $\Gamma_1$

Fig. 4.8: $L^2$-errors for 1000 iterations for the Ex. 4.3; exact input data




(a) boundary function on $\Gamma_1$, $k^* = 44$ (b) normal derivative on $\Gamma_1$, $k^* = 30$

Fig. 4.9: The exact (–) and approximate (- - -) solutions for the Ex. 4.3; noisy input data (4%). $\|u_{ex} - \tilde{u}\|_{L^2(\Gamma_1)} \approx 0.045$, $\|\frac{\partial u_{ex}}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu}\|_{L^2(\Gamma_1)} \approx 0.656$




(a) $L^2$-error for the function on $\Gamma_1$ (b) $L^2$-error for the normal derivative on $\Gamma_1$

Fig. 4.10: $L^2$-errors for 500 iterations for the Ex. 4.3; noisy input data

## 5 Conslusion

We have considered the iterative Landweber method for the Cauchy problem which requires solving two mixed well-posed Dirichlet–Neumann problems at each iteration. For this purpose, the boundary integral equation method was employed, and the mixed problem was reformulated as a system of boundary integral equations of the second kind. As demonstrated by the numerical examples above, the reconstruction of the boundary function and the normal derivative on the inner boundary achieves satisfactory accuracy and shows reasonable stability in the presence of noisy data.

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