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The accuracy estimates of the Cayley transform method for the abstract Cauchy problem

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Анотація. Одержано оцінки точності методу перетворення Келі для розв'язання задачі Коші для однорідного диференціального рівняння першого порядку з необмеженим операторним коефіцієнтом у гільбертовому просторі. У випадку скінченної (в певному сенсі) гладкості початкового вектора наш метод має степеневу швидкість збіжності, і ця швидкість автоматично залежить від диференціальних властивостей початкового вектора (тобто метод перетворення Келі є методом без насичення точності). Якщо ж початковий вектор є нескінченно гладким, то наш метод є експоненціально збіжним. Крім того, доведено непокращуваність одержаних оцінок за порядком параметра N (параметр дискретизації N характеризує кількість доданків у зображенні наближеного розв'язку).

Abstract. We obtain the accuracy estimates of the Cayley transform method for solving the initial value problem for a homogeneous first-order differential equation with an unbounded operator coefficient in a Hilbert space. In the case of finite (in some sense) smoothness of the initial vector, our method has a power-law rate of convergence and, moreover, the rate automatically depends on this regularity (i.e. the Cayley transform method is a method without saturation of accuracy). If the initial vector is infinitely smooth, then our method is exponentially convergent. In addition, we substantiate that the estimates are unimprovable in the order of N (the discretization parameter N characterizes the number of summands in the partial sum of the approximate solution).

1 Introduction

One of the effective methods for solving approximately initial value and boundary value problems in Hilbert and Banach spaces is the Cayley transform method (see, e.g. [1, 4]).

Let H be a Hilbert space with the inner product (u, v) and the associated norm $\|u\| = \sqrt{(u, u)}$. We consider in H the Cauchy problem

$$\begin{gathered}\frac{dx(t)}{dt} + Ax(t) = 0\,, \quad t > 0\,,\\ x(0) = x_0\,,\end{gathered} \tag{1.1}$$

where A is a self adjoint positive definite operator $A = A^* \geqslant \lambda_0 I,\ \lambda_0 > 0$, with the dense domain $D(A) \subset H$.

We will recall now some important results obtained in [1, 4]. It is proved that under the assumption $x_0 \in D(A^\sigma),\ \sigma > 1$, which actually means finite smoothness of the initial vector x_0, the solution $x(t) = e^{-tA}x_0$ of problem (1.1) can be represented by the series

$$x(t) = e^{-\gamma t}\sum_{p=0}^{\infty}(-1)^p L_p^{(0)}(2\gamma t)T_\gamma^p(I + T_\gamma)x_0\,, \tag{1.2}$$

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where $\gamma > 0$ is an arbitrary number, $L_p^{(0)}(t)$ is the Laguerre polynomial [3], I is an identity operator, T_γ is the Cayley transformation of the operator A:

$$T_\gamma = (\gamma I + A)^{-1}(\gamma I - A). \tag{1.3}$$

The continuous problem (1.1) generates the discrete problem

$$y_{\gamma,\, n+1} = T_\gamma y_{\gamma,\, n}, \quad n = 0, 1, \ldots, \quad y_{\gamma,0} = x_0 \,.$$

The solution of the latter can be written through the discrete semigroup T_γ^n as follows:

$$y_{\gamma,n} = T_\gamma^n y_{\gamma,0} \equiv T_\gamma^n x_0, \quad n = 0, 1, \ldots \,. \tag{1.4}$$

Note that series (1.2) converges uniformly (in H) on the closed half-line $0 \leqslant t < +\infty$ for $x_0 \in D(A^\sigma)$ with $\sigma > 0$. This implies that for $\sigma > 0$ the sum $x(t)$ of series (1.2) is a continuous function for all $t \geqslant 0$ and can be considered a generalized solution of problem (1.1).

The partial sum of series (1.2) was taken as the approximate solution of problem (1.1):

$$x_N(t) = e^{-\gamma t} \sum_{p=0}^{N} (-1)^p L_p^{(0)}(2\gamma t) T_\gamma^p (I + T_\gamma) x_0 \,. \tag{1.5}$$

For the accuracy of (1.5), the following estimate holds true [1,4]:

$$\|x(t) - x_N(t)\| \leqslant \frac{C}{N^\sigma} \|A^\sigma x_0\|, \quad t \geqslant 0 \,, \tag{1.6}$$

with a positive constant C independent of N and x_0.

The study of the Cayley transform method (1.5) was further continued in [10]. There was obtained the estimate similar to (1.6) but of the integral type, namely:

$$z_N \equiv \left\{ \int\limits_0^{+\infty} \|x(t) - x_N(t)\|^2 dt \right\}^{1/2} \leqslant \frac{C}{N^{\sigma+1/2}} \|A^\sigma x_0\|, \tag{1.7}$$

where $\sigma > 0$, and the constant $C > 0$ is independent of N and x_0.

It was also established that estimate (1.7) is unimprovable (up to logarithm) in the order of N. Namely, there exists an operator A and an initial vector x_0 such that under the assumptions $\lambda_0 = \gamma \leqslant 1$, $\sigma > 0$, $2\gamma \leqslant 1 + \sigma$ the following estimate holds true:

$$z_N^2 \geqslant \frac{4^{-11/2-\sigma}}{(N + \sigma/2)^{2\sigma+1} \ln(2N)} \,.$$

In [9] the error estimate similar to (1.7) is obtained in a different way and the constant $C > 0$ is given explicitly. Namely, if the operator A and the initial vector x_0 in problem (1.1) satisfy the conditions

$$A = A^* \geqslant \lambda_0 I, \ \lambda_0 > 0\,, \quad x_0 \in D(A^\sigma), \ \sigma > 0\,, \quad 0 < \gamma \leqslant \lambda_0\,,$$

then the accuracy of the approximate solution (1.5) is characterized by the estimate (1.7) with the positive constant $C = \dfrac{(1+\sigma)^{2(1+\sigma)}}{(2\sigma+1)(2\gamma)^{2\sigma+1}}$ independent of N and x_0 and for all positive integer numbers $N + 1 \geqslant \dfrac{\lambda_0(1+\sigma)}{2\gamma}$. Moreover, it is also shown that estimate (1.7) is unimprovable in the order of N (without logarithm), i.e. for some operator A and some initial vector x_0 the convergence rate of method (1.5) is estimated from below in the same way as in (1.7):

$$z_N^2 \geqslant \frac{C}{(N+1)^{2\sigma+1}}$$

for all $N \geqslant N^*$ with some positive integer number N^* and the constant $C > 0$ independent of N.

The aim of the present paper is to obtain some new estimates for the accuracy of the approximate solution (1.5) and study their unimprovability in the order of N.

2 Finite smoothness of the initial vector

We obtain here another integral estimate (similar to (1.7) but with a certain weight function) and substantiate its unimprovability in the order of the discretization parameter N. First we prove the following proposition.

Theorem 2.1. *Let the operator A and the initial vector x_0 in problem (1.1) satisfy the conditions*

$$A = A^* \geqslant \lambda_0 I,\ \lambda_0 > 0\,, \quad x_0 \in D(A^\sigma),\ \sigma > 0\,, \quad 0 < \gamma \leqslant \lambda_0\,.$$

Then the accuracy of the approximate solution (1.5) is characterized by the estimate

$$z_N \equiv \left\{ \int\limits_0^{+\infty} t^{-1} \|x(t) - x_N(t)\|^2 dt \right\}^{1/2} \leqslant \frac{C}{N^\sigma} \|A^\sigma x_0\| \tag{2.1}$$

for all positive integer numbers $N \geqslant \max\left\{\dfrac{\lambda_0 \sigma}{2\gamma};\ \sigma\right\} - 1$, where $C = \dfrac{\sigma^{2\sigma-1}}{e^\sigma 2^{2\sigma+1}\gamma^{2\sigma}}$ is a positive constant independent of N and x_0.

Proof. Making use of the well-known properties of the Laguerre polynomials [3], we transform the solution $x(t)$ as follows:

$$\begin{aligned} x(t) &= e^{-\gamma t} \sum_{n=0}^{\infty} (-1)^n L_n^{(0)}(2\gamma t)\,(y_{\gamma,n} + y_{\gamma,n+1}) = \\ &= e^{-\gamma t} y_{\gamma,0} + e^{-\gamma t} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\gamma t}{n} L_{n-1}^{(1)}(2\gamma t) y_{\gamma,n} \end{aligned} \tag{2.2}$$

with T_γ and $y_{\gamma,n}$ defined in (1.3) and (1.4) respectively. Then we have

$$\begin{aligned} &\|x(t) - x_N(t)\|^2 = \\ &= e^{-2\gamma t}(2\gamma t)^2 \sum_{i,j=N+1}^{\infty} \frac{(-1)^{i+j}}{ij} L_{i-1}^{(1)}(2\gamma t) L_{j-1}^{(1)}(2\gamma t)\,(y_{\gamma,i},\ y_{\gamma,j}). \end{aligned}$$

Taking into account the orthogonality condition for the Laguerre polynomials [3] and denoting by E_λ the resolution of the identity for the operator A, we get the formula

$$\begin{aligned} z_N^2 &= \int\limits_0^\infty t^{-1} \|x(t) - x_N(t)\|^2 dt = \sum_{n=N+1}^{\infty} \frac{1}{n} \|y_{\gamma,n}\|^2 = \\ &= \sum_{n=N+1}^{\infty} \frac{1}{n} \int\limits_{\lambda_0}^{\infty} \left(\frac{\gamma - \lambda}{\gamma + \lambda} \right)^{2n} \frac{1}{\lambda^{2\sigma}} d\,\|E_\lambda A^\sigma x_0\|^2. \end{aligned} \tag{2.3}$$

Next, we consider the function

$$f(\lambda) = \left(\frac{\lambda - \gamma}{\lambda + \gamma} \right)^{2n} \frac{1}{\lambda^{2\sigma}}\,, \quad \lambda > 0.$$

For the parameter $\gamma > 0$ such that $\gamma \leqslant \lambda_0$ and for all integer $n \geqslant \lambda_0\sigma/(2\gamma)$, the derivative

$$f'(\lambda) = \frac{-2(\lambda - \gamma)^{2n-1}}{\lambda^{2\sigma+1}(\lambda + \sigma)^{2n+1}} \left(\sigma\lambda^2 - 2n\gamma\lambda - \sigma\gamma^2 \right)$$

takes its maximum at the point $\lambda = \gamma\big(n + \sqrt{n^2+\sigma^2}\,\big)/\sigma$. Making some easy calculations, we come to the inequality

$$\max_{\lambda \geqslant \lambda_0} f(\lambda) = f\Big(\frac{\gamma\big(n+\sqrt{n^2+\sigma^2}\,\big)}{\sigma}\Big) = \tag{2.4}$$

$$= \left(\frac{n+\sqrt{n^2+\sigma^2}-\sigma}{n+\sqrt{n^2+\sigma^2}+\sigma}\right)^{2n} \frac{\sigma^{2\sigma}}{\gamma^{2\sigma}\big(n+\sqrt{n^2+\sigma^2}\,\big)^{2\sigma}} \leqslant \frac{\sigma^{2\sigma}}{e^{\sigma}\gamma^{2\sigma}(2n)^{2\sigma}}$$

for all integer $n \geqslant \max\{\lambda_0\sigma/(2\gamma);\ \sigma\}$. Then representation (2.3) leads to the estimate

$$z_N^2 \leqslant \frac{\sigma^{2\sigma}}{e^{\sigma}\gamma^{2\sigma}2^{2\sigma}}\|A^{\sigma}x_0\|^2 \sum_{n=N+1}^{\infty} \frac{1}{n^{2\sigma+1}} \leqslant$$

$$\leqslant \frac{\sigma^{2\sigma}}{e^{\sigma}\gamma^{2\sigma}2^{2\sigma}}||A^{\sigma}x_0||^2 \int\limits_N^{+\infty} \frac{ds}{s^{2\sigma+1}} = \frac{\sigma^{2\sigma-1}}{e^{\sigma}\gamma^{2\sigma}2^{2\sigma+1}N^{2\sigma}}||A^{\sigma}x_0||^2$$

if $N+1 \geqslant \max\{\lambda_0\sigma/(2\gamma);\ \sigma\}$. The theorem is proved. $\square$

In the next statement, we address the issue of unimprovablility of estimate (2.1).

Theorem 2.2. *Estimate (2.1) is unimprovable in the order of N.*

Proof. It suffices to present an operator A and an initial vector x_0 such that the accuracy of the Cayley transform method (1.5) is estimated from below in the order of N in the same way as in (2.1). Let the operator A have a countable set of eigenvalues

$$0 < \lambda_0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots$$

with the corresponding eigenvectors e_k, $k = 0, 1, \ldots$, forming an orthonormal basis in H: $(e_j, e_k) = \delta_{jk}$, $j,k = 0,1,\ldots$, where δ_{jk} is the Kronecker delta symbol. Then

$$y_{\gamma,n} = T_{\gamma}^n x_0 = \sum_{i=0}^{\infty} \left(\frac{\gamma-\lambda_i}{\gamma+\lambda_i}\right)^n \frac{1}{\lambda_i^{\sigma}}(A^{\sigma}x_0,\ e_i)e_i\,,$$

and the error squared takes the form

$$z_N^2 = \sum_{n=N+1}^{\infty} \frac{1}{n}\|y_{\gamma,n}\|^2 = \sum_{n=N+1}^{\infty} \frac{1}{n} \sum_{i=0}^{\infty} \left(\frac{\gamma-\lambda_i}{\gamma+\lambda_i}\right)^{2n} \frac{1}{\lambda_i^{2\sigma}}(A^{\sigma}x_0, e_i)^2,$$

$$N = 0, 1, \ldots, \quad \gamma > 0.$$

For $0 < \gamma < 1$ and $\lambda_i = i$, $i \in \mathbb{N}$, we have

$$\begin{aligned} z_N^2 &\geqslant \sum_{n=N+1}^{2N} \frac{1}{n} \sum_{i=1}^{4N} \left(\frac{i-\gamma}{i+\gamma}\right)^{2n} \frac{1}{\lambda_i^{2\sigma}}(A^{\sigma}x_0, e_i)^2 \geqslant \\ &\geqslant \frac{1}{2(4N)^{2\sigma}} \sum_{i=1}^{4N} \left(\frac{i-\gamma}{i+\gamma}\right)^{4N} (A^{\sigma}x_0, e_i)^2. \end{aligned} \tag{2.5}$$

Now we will show that for $\gamma = \frac{1}{4N}$ and $(A^{\sigma}x_0, e_i)^2 = \frac{1}{i^2}$, $i \in \mathbb{N}$, the limit

$$\lim_{N\to\infty} \sum_{i=1}^{4N} \frac{1}{i^2} \left(\frac{i-\frac{1}{4N}}{i+\frac{1}{4N}}\right)^{4N}$$

exists and is positive. It suffices to verify that the sequence

$$S_N = \sum_{i=1}^{N} \frac{1}{i^2} \left(\frac{i - \frac{1}{N}}{i + \frac{1}{N}} \right)^N, \quad N \in \mathbb{N},$$

is increasing and bounded. Indeed, since

$$S_N = \sum_{i=1}^{N} \frac{1}{i^2} \left(\frac{i - \frac{1}{N}}{i + \frac{1}{N}} \right)^N < \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} = 1.644934068\ldots$$

(more exactly,

$$S_N = \sum_{i=1}^{N} \frac{1}{i^2} \left(\frac{i - \frac{1}{N}}{i + \frac{1}{N}} \right)^N < \sum_{i=1}^{\infty} \frac{e^{-2/i}}{i^2} = 0.5012176777\ldots),$$

the sequence (S_N) of positive numbers is bounded from above. Next, we consider the difference

$$S_{N+1} - S_N = \sum_{i=1}^{N} \frac{1}{(i+1)^2} \left\{ \left(\frac{i - \frac{1}{N+1}}{i + \frac{1}{N+1}} \right)^{N+1} - \left(\frac{i - \frac{1}{N}}{i + \frac{1}{N}} \right)^N \right\} + \\ + \frac{1}{(N+1)^2} \left(\frac{N+1 - \frac{1}{N+1}}{N+1 + \frac{1}{N+1}} \right)^{N+1}$$

and prove that it is positive for all $N \in \mathbb{N}$. To this end, we introduce the auxiliary function

$$g(N) = \left(\frac{i - \frac{1}{N}}{i + \frac{1}{N}} \right)^N, \quad N > 0,$$

and find its derivative

$$g'(N) = \left(\frac{iN - 1}{iN + 1} \right)^N \left[\ln \frac{iN - 1}{iN + 1} + \frac{2iN}{(iN)^2 - 1} \right].$$

To define the sign of the expression in the brackets, we denote $(iN)^{-1} = x$ and consider the function

$$\varphi(x) = \ln \frac{1 - x}{1 + x} + \frac{2x}{1 - x^2}, \quad x \in [0; 1).$$

Since

$$\varphi'(x) = \frac{4x^2}{(1 - x^2)^2}$$

for all $x \in (0; 1)$, the function $\varphi(x)$ strictly increases on the interval $[0; 1)$, and due to the equality $\varphi(0) = 0$ we have $\varphi(x) > 0$ for all $x \in (0; 1)$. It means that $g'(N) > 0$ for all $N > 0$ and, therefore, the function $g(N)$ strictly increases. This yields $S_{N+1} - S_N > 0$ for all $N \in \mathbb{N}$ and proves that the sequence (S_N) is strictly increasing.

Since the sequence (S_N) is monotonic and bounded, it is convergent. It is obvious that $\lim\limits_{N \to \infty} S_N$ is a positive number. Then there exist $N^* \in \mathbb{N}$ and a positive constant $\tilde{C}$ such that $S_N \geqslant \tilde{C}$ for all $N \geqslant N^*$. Therefore, inequality (2.5) leads to the estimate

$$z_N^2 \geqslant \frac{C}{N^{2\sigma}}$$

for all $N \geqslant N^*$ with a constant C independent of N. This completes the proof of the theorem. $\square$

Note that in addition to umiprovability in the order of N, estimate (2.1) indicates the initial effect of the Cayley transform method (1.4) in the sense of V. Makarov (see, e.g. [8]). Indeed, due to representation of the exact solution (2.2), estimate (2.4) and the relation $L^{(1)}_{n-1}(0) = n$ [3], we have

$$\lim_{t\to 0}\frac{\|x(t)-x_N(t)\|}{t} =$$
$$= \sum_{n=N+1}^{\infty}(-1)^{n+1}2\gamma\int\limits_{\lambda_0}^{+\infty}\left(\frac{\gamma-\lambda}{\gamma+\lambda}\right)^n\frac{1}{\lambda^\sigma}d\,E_\lambda A^\sigma x_0 \leqslant$$
$$\leqslant \sum_{n=N+1}^{\infty}\frac{\sigma^\sigma}{e^{\sigma/2}(2\gamma)^{\sigma-1}n^\sigma}\|A^\sigma x_0\| \leqslant$$
$$\leqslant \frac{\sigma^\sigma}{e^{\sigma/2}(2\gamma)^{\sigma-1}}\|A^\sigma x_0\|\int\limits_N^{+\infty}\frac{ds}{s^\sigma} \leqslant \frac{\sigma^\sigma}{e^{\sigma/2}(2\gamma)^{\sigma-1}(\sigma-1)N^{\sigma-1}}\|A^\sigma x_0\|$$

for all positive integer numbers $N+1 \geqslant \max\{\lambda_0\sigma/(2\gamma);\ \sigma\}$ and $\sigma > 1$. We can observe that the use of the initial effect requires more assumptions for the smoothness of the initial vector $x_0 \in D(A^\sigma)$, $\sigma > 1$. If the initial effect is not used, then it is enough that $\sigma > 0$. A similar situation is inherent in the finite-difference method for solving boundary-value problems for parabolic equation (see details in papers [6, 7, 11, 12] to mention a few).

3 Infinite smoothness of the initial vector

Let E be a Banach space with the norm $\|u\|$. We denote by $\mathfrak{R}(A,\mu,L)$ the set of elements $x \in E$ such that

$$\|x\|_{\mathfrak{R}(A,\,\mu,\,L)} = \sup_{0\leqslant n<\infty}\frac{\|A^n x\|}{L^n M_n} < \infty\,,$$

where $0 < L < +\infty$ and $\mu = (M_n)_{n=0}^{\infty}$ is a non-decreasing sequence of positive numbers. As shown in [5], $\mathfrak{R}(A,\mu,L)$ is continuously embedded in E.

It was proved in [10] that in the case of the discrete spectrum of the operator A and infinite smoothness of the initial vector x_0, i.e. $x_0 \in D(A^n)$ for all $N \in \mathbb{N}$, the Cayley transform method (1.5) is exponentially convergent and its accuracy is characterized by the estimate

$$z_N \equiv \left\{\int\limits_0^{+\infty}\|x(t)-x_N(t)\|^2 dt\right\}^{1/2} \leqslant$$
$$\leqslant \frac{\exp\left\{-(N+1)\ln\left(1+\sqrt{2}\right)\right\}}{\left(\gamma\sqrt{2}\,(2+\sqrt{2})\right)^{1/2}}\|x_0\|_{\mathfrak{R}(A,\,(1),\,\lambda_0)}\,,$$

which is unimprovable in the order of N.

We will now show that a similar estimate can be obtained without assuming the discreteness of the spectrum of the operator A.

Theorem 3.1. *Let the operator A and the initial vector x_0 in problem (1.1) satisfy the conditions*

$$A = A^* \geqslant \lambda_0 I,\ \lambda_0 > 0\,,\quad x_0 \in \mathfrak{R}\,(A,\,(1),\,\lambda_0)\,,\quad \gamma = \frac{\lambda_0}{1+\sqrt{2}}\,.$$

Then the accuracy of the approximate solution (1.5) is characterized by the estimate

$$z_N \equiv \left\{ \int\limits_0^{+\infty} ||x(t) - x_N(t)||^2 dt \right\}^{1/2} \leqslant \tag{3.1}$$

$$\leqslant \left(\frac{1+\sqrt{2}}{2\gamma} \right)^{1/2} \exp\left\{ -(N+2)\ln(1+\sqrt{2}) \right\} \|x_0\|_{\mathfrak{R}(A,\,(1),\,\lambda_0)} \,,$$

$$N = 0, 1, \ldots ,$$

which is unimprovable in the order of N.

Proof. The series

$$x(t) - x_N(t) = e^{-\gamma t} \sum_{p=N+1}^{\infty} (-1)^p L_p^{(0)}(2\gamma t)\, (y_{\gamma,p} + y_{\gamma,p+1})\,, \quad t \geqslant 0\,,$$

leads to the relation

$$\|x(t) - x_N(t)\|^2 = \big(x(t) - x_N(t),\, x(t) - x_N(t)\big) = \tag{3.2}$$

$$= \Bigg(e^{-\gamma t} \sum_{p=N+1}^{\infty} (-1)^p L_p^{(0)}(2\gamma t)\, (y_{\gamma,p} + y_{\gamma,p+1}),$$

$$e^{-\gamma t} \sum_{p=N+1}^{\infty} (-1)^p L_p^{(0)}(2\gamma t)\, (y_{\gamma,p} + y_{\gamma,p+1}) \Bigg) =$$

$$= e^{-2\gamma t} \sum_{j,\,k=N+1}^{\infty} (-1)^{j+k} L_j^{(0)}(2\gamma t) L_k^{(0)}(2\gamma t) \big(y_{\gamma,j} + y_{\gamma,j+1},\, y_{\gamma,k} + y_{\gamma,k+1}\big).$$

The generalized Laguerre polynomial $L_n^{(\alpha)}(t)$, $\alpha > -1$, satisfy the orthogonality condition with the weight function [3]:

$$\int\limits_0^{+\infty} e^{-t} t^{\alpha} L_j^{(\alpha)}(t) L_k^{(\alpha)}(t) dt = \frac{\Gamma(\alpha + k + 1)}{k!} \delta_{jk}\,,$$

$$j, k = 0, 1, \ldots, \quad \alpha > -1\,,$$

where δ_{jk} is the Kronecker delta symbol. In particular, for $\alpha = 0$ we have

$$\int\limits_0^{+\infty} e^{-2\gamma t} L_j^{(0)}(2\gamma t) L_k^{(0)}(2\gamma t) dt = \frac{1}{2\gamma} \frac{\Gamma(k+1)}{k!} \delta_{jk} = \frac{1}{2\gamma} \delta_{jk}\,, \quad j, k = 0, 1, \ldots$$

Taking this formula into account while integrating relation (3.2), we obtain

$$z_N^2 \equiv \int\limits_0^{+\infty} \|x(t) - x_N(t)\|^2 dt = \tag{3.3}$$

$$= \sum_{j,\,k=N+1}^{\infty} (-1)^{j+k} \int\limits_0^{+\infty} e^{-2\gamma t} L_j^{(0)}(2\gamma t) L_k^{(0)}(2\gamma t) dt \big(y_{\gamma,j} + y_{\gamma,j+1},\, y_{\gamma,k} + y_{\gamma,k+1}\big) =$$

$$= \frac{1}{2\gamma} \sum_{k=N+1}^{\infty} (y_{\gamma,k} + y_{\gamma,k+1}, y_{\gamma,k} + y_{\gamma,k+1}) = \frac{1}{2\gamma} \sum_{k=N+1}^{\infty} \|y_{\gamma,k} + y_{\gamma,k+1}\|^2.$$

Due to (1.3) and (1.4), we get

$$\begin{aligned} y_{\gamma,k} + y_{\gamma,k+1} = T_\gamma^k (I + T_\gamma) x_0 = \\ = (\gamma I + A)^{-k} (\gamma I - A)^k \big(I + (\gamma I + A)^{-1} (\gamma I - A)\big) x_0 = \\ = (\gamma I + A)^{-k} (\gamma I - A)^k 2\gamma (\gamma I + A)^{-1} A^{-\sigma} A^{\sigma} x_0 \,, \quad \sigma > 0. \end{aligned} \tag{3.4}$$

Then for all $N \in \mathbb{N}$, formula (3.4) can be written as follows:

$$\begin{aligned} y_{\gamma,k} + y_{\gamma,k+1} = (\gamma I + A)^{-k} (\gamma I - A)^k 2\gamma (\gamma I + A)^{-1} A^{-n} A^n x_0 = \\ = \int_{\lambda_0}^{+\infty} \left(\frac{\gamma - \lambda}{\gamma + \lambda}\right)^k \frac{2\gamma}{\gamma + \lambda} \frac{\lambda_0^n}{\lambda^n} dE_\lambda \frac{A^n x_0}{\lambda_0^n} \,, \end{aligned}$$

which gives the representation

$$\|y_{\gamma,k} + y_{\gamma,k+1}\|^2 = \int_{\lambda_0}^{+\infty} \left(\frac{\gamma - \lambda}{\gamma + \lambda}\right)^{2k} \frac{(2\gamma)^2}{(\gamma + \lambda)^2} \frac{\lambda_0^{2n}}{\lambda^{2n}} \, d\left\|E_\lambda \frac{A^n x_0}{\lambda_0^n}\right\|^2.$$

In particular, for $n = k$ we have the relation

$$\|y_{\gamma,k} + y_{\gamma,k+1}\|^2 = \int_{\lambda_0}^{+\infty} \left(\frac{\gamma - \lambda}{\gamma + \lambda}\right)^{2k} \frac{(2\gamma)^2}{(\gamma + \lambda)^2} \left(\frac{\lambda_0}{\lambda}\right)^{2k} d\left\|E_\lambda \frac{A^k x_0}{\lambda_0^k}\right\|^2,$$

and formula (3.3) turns into the following:

$$z_N^2 = 2\gamma \sum_{k=N+1}^{\infty} \int_{\lambda_0}^{+\infty} \left(\frac{\gamma - \lambda}{\gamma + \lambda}\right)^{2k} \frac{1}{(\gamma + \lambda)^2} \left(\frac{\lambda_0}{\lambda}\right)^{2k} d\left\|E_\lambda \frac{A^k x_0}{\lambda_0^k}\right\|^2. \tag{3.5}$$

Next, we consider the function

$$\varphi(\lambda) = \frac{1}{(\lambda + \gamma)^{k+1}} \left(\frac{\lambda - \gamma}{\lambda}\right)^k$$

and find the derivative

$$\varphi'(\lambda) = \frac{-(\lambda - \gamma)^{k-1}}{(\lambda + \gamma)^{k+2} \lambda^{k+1}} \big((k+1)\lambda^2 - \gamma(2k+1)\lambda - k\gamma^2\big) .$$

For all $\gamma > 0$ and $k \in \mathbb{N}$, the roots λ_1, λ_2 of the latter satisfy the inequality

$$\begin{aligned} \lambda_1 = \gamma < \lambda_2 = \frac{\gamma(2k+1) + \gamma 2\sqrt{2}\sqrt{(k+1/2)^2 - 1/8}}{2(k+1)} < \\ < \frac{\gamma(2k+1)(1+\sqrt{2})}{2k+2} < \gamma(1+\sqrt{2}) \,. \end{aligned}$$

If $\gamma = \dfrac{\lambda_0}{1+\sqrt{2}}$, then the function $\varphi(\lambda)$ strictly decreases on the half-line $[\lambda_0; +\infty)$, and relation (3.5) finally gives the estimate

$$z_N^2 \leqslant 2\gamma \sum_{k=N+1}^{\infty} \int\limits_{\lambda_0}^{+\infty} \left(\frac{\gamma-\lambda_0}{\gamma+\lambda_0}\right)^{2k} \frac{1}{(\gamma+\lambda_0)^2} \left(\frac{\lambda_0}{\lambda_0}\right)^{2k} d\left\|E_\lambda \frac{A^k x_0}{\lambda_0^k}\right\|^2 \leqslant$$
$$\leqslant \frac{1}{\gamma} \|x_0\|^2_{\mathfrak{R}(A,\,(1),\,\lambda_0)} \sum_{k=N+1}^{\infty} \left(\frac{1}{1+\sqrt{2}}\right)^{2(k+1)} =$$
$$= \frac{1+\sqrt{2}}{2\gamma} \big(1+\sqrt{2}\big)^{-2(N+2)} \|x_0\|^2_{\mathfrak{R}(A,\,(1),\,\lambda_0)}.$$

As shown in [10], for the operator A with the discrete spectrum

$$\lambda_0 = \gamma(1+\sqrt{2}) \leqslant \lambda_1 \leqslant \ldots$$

and for the initial vector $x_0 \in \mathfrak{R}\,(A,(1),\lambda_0)$ the following estimate holds true:

$$z_N \geqslant \frac{|(x_0,e_0)|}{\gamma^{1/2}} \exp\left\{-(N+2)\ln\left(1+\sqrt{2}\right)\right\}, \quad N = 0,1,\ldots,$$

where e_0 is a normalized eigenvector corresponding to the eigenvalue λ_0. This inequality indicates that estimate (3.1) is unimprovable in the order of N. The theorem is proved. $\square$

4 Conclusion

The error estimate obtained in Theorem 2.1 indicates that for the *finitely* smooth initial vector x_0 (in terms of belonging to the domain of a certain power of the operator A) the Cayley transform method (1.5) has a power-law rate of convergence with respect of the discretization parameter N (here N is the number of terms in the partial sum representing the approximate solution). Moreover, method (1.5) is a method without saturation of accuracy in the sense of K. Babenko [2], i.e. the better differential properties of the initial vector, the higher the rate of convergence. In addition, Theorem 2.2 substantiates that this estimate is unimprovable in the order of N. The accuracy of the approximate solution (1.5) for the *infinitely* smooth initial vector (in some sense) is studied in Theorem 3.1. The error estimate indicates the exponential rate of convergence and, moreover, is unimprovable with respect of the parameter N.

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References

1. Arov, D.Z., Gavrilyuk, I.P., Makarov, V.L.: Representation and approximation of solutions of initial value problems for differential equations in Hilbert space based on the Cayley transform. Elliptic and Parabolic Problems (Pont-à-Mousson, 1994). Pitman Res. Notes Math. Ser., 325, Longman Sci. Tech., Harlow 40-50 (1995)
2. Babenko, K.I.: The Foundations of Numerical Methods. Reguliarnaia i khaoticheskaia dinamika, Moskva–Izhevsk (2002)
3. Bateman, H., Erdélyi, A.: Higher Transcendental Functions, vol. 2, McGraw-Hill, New York (1953)

4. Gavrilyuk, I.P., Makarov, V.L.: The Cayley transform and the solution of an initial value problem for a first order differential equation with an unbounded operator coefficient in Hilbert space. Numerical Functional Analysis and Optimization. **15** (5-6), 583-598 (1994). doi: 10.1080/01630569408816582
5. Gorbachuk, V.I., Gorbachuk, M.L.: Limit Problems for Differential Operator Equations. Naukova dumka, Kyiv (1984)
6. Makarov, VL., Demkiv, L.I.: Accuracy estimates of difference schemes for quasilinear parabolic equations taking into account the initial-boundary effect. Computational Methods in Applied Mathematics. **3** (4), 579-595 (2003)
7. Makarov, V.L., Demkiv L.I.: Accuracy estimates of finite difference schemes for parabolic equations that take into account the initial-boundary effect. Reports of the National Academy of Sciences of Ukraine. **2**, 26-32 (2003)
8. Makarov, V.L., Mayko, N.V.: The weighted error estimates of the functional-discrete methods for solving boundary value problems. Naukova dumka, Kyiv (2023)
9. Makarov, V.L., Mayko, N.V., Ryabichev, V.L.: Unimprovable error estimates of the Cayley transform method for the operator exponential function in a Hilbert space. Cybernetics and Systems Analysis. **59** (6), (2023, to appear)
10. Makarov, V.L., Vasylyk, V.B., Ryabichev, V.L.: Unimprovable order-of-magnitude estimates of the rate of convergence of the Cayley transform method for approximation of an operator exponent. Cybernetics and Systems Analysis. **38** (4), 632-636 (2002). doi: 10.1023/A:1021122622419
11. Mayko, N.V.: The boundary effect in the error estimate of the finite-difference scheme for the two-dimensional heat equation. Journal of Numerical and Applied Mathematics. **3** (113), 91-106 (2013)
12. Mayko, N.V.: Error estimates of the finite-difference scheme for a one-dimensional parabolic equation with allowance for the effect of initial and boundary conditions. Cybernetics and Systems Analysis. **50**, 788-796 (2014)

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