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A fundamental sequences method with time-reduction for one-dimensional lateral Cauchy problems

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Анотація. Розроблено метод фундаментальних послідовностей для чисельного розв'язання некоректної одновимірної бічної задачі Коші для рівняння гіперболічної затухаючої хвилі, включаючи як окремий випадок рівняння параболічної теплопровідності. Застосовуючи або перетворення Лаґерра, або різницеву схему Губольта, залежну від часу бічну задачу Коші редуковано до послідовності звичайних диференціальних рівнянь (ЗДР) другого порядку зі значеннями функції та похідними, визначеними в правій кінцевій точці просторового інтервалу. Для диференціальних рівнянь будується набір фундаментальних розв'язків, який називається фундаментальною послідовністю. Розв'язок отриманих ЗДР апроксимується лінійною комбінацією елементів фундаментальної послідовності. Точки джерела розміщуються за межами інтервалу розв'язків у просторі, і шляхом розміщення точок колокації в кінцевих точках цього інтервалу отримують послідовність систем лінійних рівнянь для знаходження невідомих коефіцієнтів. Для отримання стійкого розв'язку використовується регуляризація Тіхонова. Чисельні результати як для параболічного, так і для гіперболічного випадку підтверджують ефективність запропонованого методу, включаючи випадок даних зі шумом. Представлені результати доповнюють випадок вищої розмірності, розпочатий у наших попередніх дослідженнях.

Abstract. A fundamental sequences method is derived for the numerical solution of an ill-posed one-dimensional lateral Cauchy problem for a hyperbolic damped wave equation, including as a special case the parabolic heat equation. Either the Laguerre transform or the Houbolt finite difference scheme is applied to reduce the time-dependent lateral Cauchy problem to a sequence of second-order ordinary differential equations (ODEs) with function values and derivatives specified at the right endpoint of a finite space interval. A set of fundamental solutions is constructed, termed a fundamental sequence, to the differential equations. The solution of the obtained ODEs is approximated by a linear combination of elements in the fundamental sequence. Source points are placed outside of the solution interval in space, and by collocating at the endpoints of this interval a sequence of linear equations is obtained for finding the unknown coefficients. Tikhonov regularization is used to render a stable solution to the obtained systems of linear equations. Numerical results both for the parabolic and hyperbolic case confirm the efficiency of the proposed method including noisy data. The presented results complement the higher-dimensional case initiated in our previous researches.

1 Introduction

In the works [5, 6] fundamental sequences method is developed for solving ill-posed lateral Cauchy problems for time-dependent heat and wave propagation in two-dimensional or higher spatial domains. The method is based on time-transformation using the Laguerre transform or time-discretisation using the so-called Houbolt finite difference scheme. The time-transformation or discretisation renders a sequence of elliptic Cauchy problems with fundamental solutions termed a fundamental sequence. The solution of the elliptic problems is approximated by linear combinations of the elements in the fundamental sequence, with source points placed outside of the solution

Key words: damped wave equation, heat equation, Houbolt method, Laguerre transform, lateral Cauchy problem, method of fundamental solutions, fundamental sequences method, Tikhonov regularization.

domain. Collocating on the boundary of the solution domain renders linear equations for finding unknown coefficients in the approximation. For an overview and properties of the derived fundamental sequences method, and comparison with a boundary integral approach, see the recent review [8].

One-dimensional time-dependent models are common in the literature since they can give reasonable insights of a physical problems such as string vibration problem, water or sound wave propagation, stress wave propagation in an elastic solid, etc. Hence, it is no surprise that there has been recent interest into the lateral Cauchy problem for the one-dimensional wave equation, see for example [2,4,14,28].

Therefore, in the present work, we shall further complete the fundamental sequences method, derived in [5,6], by giving details for one-dimensional spatial domains. We thus consider the problem of finding the solution u satisfying

$$\begin{cases} au''_{tt} + bu'_t - u''_{xx} = 0, & (x,t) \in (0,1) \times (0,T), \\ u(1,t) = \varphi_1(t), \;\; u'_x(1,t) = \tilde{\varphi}_1(t), & t \in (0,T), \\ u(x,0) = 0, \;\; au'_t(x,0) = 0, & x \in (0,1). \end{cases} \tag{1.1}$$

Here, $a \geq 0$, $b \geq 0$ and $a + b > 0$, and $T > 0$ is the final time; for simplicity the spatial interval $(0,1)$ is chosen. Of particular interest is to find the solution when $x = 0$, that is $u(0,t)$. For suitable compatibility and smoothness conditions on the data, there exists a unique solution, for example, in the Sobolev space $L^2(0,T; H^1(0,1)) \cap H^1(0,T; L^2(0,1))$. However, stability cannot be guaranteed, i.e., finding u satisfying (1.1) is an ill-posed problem.

In the case when $a = 0$ in (1.1), we have the Cauchy problem for the heat equation, which is a classical and well-studied inverse ill-posed problem. To provide some general references where properties and methods for that problem can be found, see for example, [7, 12, 19, 24, 26] and references therein. In the case when instead $b = 0$ in (1.1), we have the corresponding lateral Cauchy problem for the wave equation, which is considerably less studied. Uniqueness of a solution in this case is a more subtle issue, due to finite speed of propagation, than for the heat equation, and is shown to hold in the one-dimensional case in [22] and more generally in [3,4,28]. An example highlighting the instability is given in [23, Sect. 1] (for more on hyperbolic inverse problems, see [17]). In the literature, lateral one-dimensional Cauchy problems for the heat equation is sometimes termed the sideways heat equation [13] and correspondingly for the wave equation [28]. When both a and b are non-zero in (1.1) the governing equation is the classical damped wave equation, also known as the telegraph equation; for a derivation in four different physical settings, see [27], and for examples in option pricing [25], heat waves [21,32] and exact controllability [30].

For the outline of the work, in Section 2 time transformation using the Laguerre transform is discussed as well as time discretisation via the Houbolt method. The system obtained after time-reduction and consisting of second-order ordinary differential equations (ODEs) is stated. A fundamental sequences method for the obtained sequence of ODEs is derived in Section 3 together with some properties. The fundamental sequence is explicitly constructed, see Theorem 3.1. Tikhonov regularization is applied to the discretized system of algebraic equations obtained after collocation at the endpoints of the space interval. Formulas for the sought approximation to (1.1) and to the Cauchy data when $x = 0$ is given at the end of that section. In Section 4 is a note on a similar fundamental sequences method for the solution of the corresponding well-posed Dirichlet problem. Section 5 presents numerical results showing the applicability of the proposed approach both for the wave and heat equation with exact as well as noisy data.

2 Semi-discretization with respect to time

2.1 Laguerre Transformation

We search for the solution of the lateral Cauchy problem (1.1) in the form of a Fourier–Laguerre expansion

$$u(x,t) = \kappa \sum_{p=0}^{\infty} u_p(x) L_p(\kappa t), \tag{2.1}$$

where

$$u_p(x) = \int_0^\infty e^{-\kappa t} L_p(\kappa t) u(x,t)\, dt, \quad p = 0,1,2,\dots .$$

Here, L_p is the Laguerre polynomial of order p, and $\kappa > 0$ a scaling parameter. Properties of the Laguerre polynomials [1] give that for a bounded and continuously differentiable function g (these conditions can be weakened) with Fourier-Laguerre coefficients g_p, the Fourier–Laguerre coefficients g'_p of the derivative g' have the following representation

$$g'_p = -g(0) + \kappa \sum_{m=0}^{p} g_m, \quad p = 0,1,2,\dots .$$

The Fourier–Laguerre coefficients g''_p of the second derivative g'' of a twice continuously differentiable function g are given by

$$g''_p = -g'(0) - \kappa(p+1)g(0) + \kappa^2 \sum_{m=0}^{p} (p-m+1) g_m, \quad p = 0,1,2,\dots .$$

After performing straightforward calculations, taking into account the representation of the solution u by (2.1) and the formulas for derivatives of Fourier–Laguerre coefficients, we find that the functions u_p in (2.1) satisfy the following sequence of ordinary differential equations with Cauchy data

$$\begin{cases} u''_p(x) - \gamma^2 u_p(x) = \displaystyle\sum_{m=0}^{p-1} \beta_{p-m} u_m(x), & x \in (0,1), \\ u_p(1) = \varphi_{1,p}, \;\; u'_p(1) = \tilde{\varphi}_{1,p}, & p = 0,1,2,\dots . \end{cases} \tag{2.2}$$

Here, $\beta_p = a\kappa^2(p+1) + b\kappa$, $\gamma^2 = \beta_0$,

$$\varphi_{1,p} = \int_0^\infty e^{-\kappa t} L_p(\kappa t) \varphi_\ell(t)\, dt, \quad p = 0,1,2,\dots,$$

$$\tilde{\varphi}_{1,p} = \int_0^\infty e^{-\kappa t} L_p(\kappa t) \tilde{\varphi}_1(t)\, dt, \quad p = 0,1,2,\dots .$$

The numerical approximation to the solution of (1.1) is a partial sum of the series (2.1), i.e., (2.2) is solved for $p = 0,1,\dots,N$, for some integer $N > 0$.

2.2 Houbolt method

As an alternative to the Laguerre transform in time the Houbolt discretisation method [18] can be used. To apply the Houbolt method to the governing equation in (1.1), we use the equidistant grid

$$t_p = (p+3)h, \quad \text{for } p = -3,-2,\dots,N, \quad \text{with } h = \frac{T}{N+3}, \text{ and } N \in \mathbb{N}, \tag{2.3}$$

and approximate the solution u to (1.1) by the sequence

$$u_p \approx u(\cdot, t_p), \quad p = -3,-2,\dots,N.$$

Using the Houbolt scheme [15], we have following approximations:

$$u'_t(\cdot, t_p) \approx \frac{1}{6h}(11u_p - 18u_{p-1} + 9u_{p-2} - 2u_{p-3})$$

and

$$u''_{tt}(\cdot, t_p) \approx \frac{1}{h^2}(2u_p - 5u_{p-1} + 4u_{p-2} - u_{p-3}).$$

After applying the standard Euler scheme, we obtain an approximation of the first three elements of the sequence:

$$u_{-3} = u(\cdot, 0) = 0,$$
$$u_{-2} \approx u_{-3} + hu'_t(\cdot, 0) = 0,$$
$$u_{-1} \approx u_{-3} + 2hu'_t(\cdot, 0) = 0.$$

As result, we find that the functions u_p satisfy the following sequence of ordinary differential equations with Cauchy data,

$$\begin{cases} u''_p(x) - \gamma^2 u_p(x) = \sum_{m=0}^{p-1} \beta_{p-m} u_m(x), & x \in (0,1), \\ u_p(1) = \varphi_{1,p}, \;\; u'_p(1) = \tilde{\varphi}_{1,p}, & p = 0, 1, 2, \ldots, \end{cases} \tag{2.4}$$

where $\varphi_{0,p} = \varphi_0(t_p)$, $\tilde{\varphi}_{1,p} = \tilde{\varphi}_1(t_p)$, for $p = 0, 1, \ldots, N$. The coefficients are

$$\gamma^2 = \beta_0 = \frac{2a}{h^2} + \frac{11b}{6h}, \quad \beta_1 = -\frac{5a}{h^2} - \frac{3b}{h}, \quad \beta_2 = \frac{4a}{h^2} + \frac{3b}{2h}, \quad \beta_3 = -\frac{a}{h^2} - \frac{b}{3h},$$

and $\beta_4 = \ldots = \beta_{N-1} = 0$.

The sequence of ODEs (2.4) is equal to the sequence (2.2), only the values of the coefficients differ. Thus, when either the Laguerre transform or the Houbolt method is applied to (1.1), we have to solve a system of ODEs of the form (2.2).

3 A fundamental sequences method for numerical solution of the system of ODEs (2.2)

We shall present a fundamental sequences method for (2.2) and start by recalling the following.

Definition 3.1. *The sequence of functions $\{\Phi_p\}_{p=0}^N$ is denoted a fundamental sequence for the equations in* (2.2) *provided that*

$$\frac{\partial^2 \Phi_p}{\partial x^2}(x, y) - \sum_{m=0}^{p} \beta_{p-m} \Phi_m(x, y) = \delta(x - y), \quad x, y \in \mathbb{R}, \quad x \neq y. \tag{3.1}$$

When $p = 0$, we have the one-dimensional modified Helmholtz equation. It can be directly verified that $\Phi_0(x, y) = \frac{1}{2\gamma} e^{-\gamma|x-y|}$ satisfies (3.1) when $p = 0$ (recall that $\gamma^2 = \beta_0$). Thus, it makes sense to search for Φ_p in the form $e^{-\gamma|x-y|} v_p(|x - y|)$ for a suitable polynomial v_p. Following [9,10], we use the polynomials v_p defined by

$$v_p(r) = \sum_{m=0}^{p} a_{p,m} r^m \tag{3.2}$$

with

$$a_{p,0} = 1, \quad p = 0, 1, \ldots, N,$$
$$a_{p,p} = -\frac{1}{2\gamma p}\, \beta_1 a_{p-1,p-1}, \quad p = 1, 2, \ldots, N$$

and

$$a_{p,k} = \frac{1}{2\gamma k} \left\{ k(k+1) a_{p,k+1} - \sum_{m=k-1}^{p-1} \beta_{p-m} a_{m,k-1} \right\}, \quad k = p - 1, \ldots, 1,$$

for $p = 2, \ldots, N$.

Straightforward calculations show that the polynomials (3.2) satisfy the following sequence of ODEs

$$v_p'' - 2\gamma v_p' = \sum_{m=0}^{p-1} \beta_{p-m} v_m.$$

Hence, we obtain the following new result.

Theorem 3.1. *The sequence of functions* $\{\Phi_p\}_{p=0}^N$ *with*

$$\Phi_p(x, y) = e^{-\gamma|x-y|} v_p(|x - y|), \quad x \neq y$$

is a fundamental sequence of (2.2) *in the sense of* Definition 3.1.

Similar to [31, Prop. 2] one can show that the obtained fundamental sequence is a fundamental set for the governing ordinary differential equations in (2.2). Thus, it makes sense to search for an approximation to (2.2) in the form

$$u_p(x) \approx \tilde{u}_p(x) = \sum_{m=0}^{p} \left(\alpha_{1,m}\Phi_{p-m}(x, y_1) + \alpha_{2,m}\Phi_{p-m}(x, y_2)\right), \ x \in (0, 1) \tag{3.3}$$

with unknown coefficients $\alpha_{1,p}$ and $\alpha_{2,p}$, with given numbers y_1 and y_2 outside the solution domain $(0, 1)$, that is $y_1 < 0$ and $y_2 > 1$. Collocating to match the boundary data in (2.2), we generate a recurrent sequence of linear systems to find the coefficients $\alpha_{1,p}$ and $\alpha_{2,p}$,

$$\begin{cases} \alpha_{1,p}\Phi_0(1, y_1) + \alpha_{2,p}\Phi_0(1, y_2) = \varphi_{1,p} - \displaystyle\sum_{m=0}^{p-1} \sum_{\ell=1}^{2} \alpha_{\ell,m}\Phi_{p-m}(1, y_\ell), \\ \alpha_{1,p}\Phi'_{0,x}(1, y_1) + \alpha_{2,p}\Phi'_{0,x}(1, y_2) = \tilde{\varphi}_{1,p} - \displaystyle\sum_{m=0}^{p-1} \sum_{\ell=1}^{2} \alpha_{\ell,m}\Phi'_{p-m,x}(1, y_\ell), \end{cases} \tag{3.4}$$

for $p = 0, 1, \ldots, N$, where derivatives of elements from fundamental sequence are

$$\Phi'_{p,x}(x, y) = \frac{x - y}{|x - y|} e^{-\gamma|x-y|}(-\gamma v_p(|x - y|) + \tilde{v}_p(|x - y|)), \quad x \neq y$$

and

$$\tilde{v}_p(r) = \sum_{m=1}^{p} m a_{p,m} r^{m-1}.$$

The linear systems (3.4) are ill-conditioned, therefore, to obtain stable solutions, we apply standard Tikhonov regularization with a regularization parameter chosen according to the heuristic L-curve rule [16]. The regularization parameter is chosen only once, because the matrix is the same in (3.4).

In total, the solution of the inverse problem (1.1), obtained using the Laguerre transformation, is computed by

$$u(x, t) \approx u_N(x, t) = \kappa \sum_{p=0}^{N} \tilde{u}_p(x) L_p(\kappa t), \tag{3.5}$$

with $\tilde{u}_p$ generated from (3.3). Moreover, the space derivative of the function is obtained from (3.5) via

$$u'_x(x, t) \approx u'_{N,x}(x, t) = \kappa \sum_{p=0}^{N} L_p(\kappa t) \sum_{m=0}^{p} \sum_{\ell=1}^{2} \alpha_{\ell,m}\Phi'_{p-m,x}(x, y_\ell). \tag{3.6}$$

In case of the time discretization via the Houbolt method, the solution of the inverse problem (1.1) is computed at the mesh points t_p, $p = 0, \dots, N$ given in (2.3) by

$$u(x, t_p) \approx \tilde{u}_p(x) \tag{3.7}$$

with $\tilde{u}_p$ from (3.3), and the space derivative of the solution is obtained by

$$u'_x(x, t_p) \approx \sum_{m=0}^{p} \sum_{\ell=1}^{2} \alpha_{\ell,m} \Phi'_{p-m,x}(x, y_\ell).$$

Note that as derived in Section 2 the values of the constants $\beta_0, \dots, \beta_N$ are different for the Laguerre and Houbolt methods. These values enter into the construction of the fundamental sequence $\{\Phi_p\}_{p=0}^{N}$ as can be seen both from Definition 3.1 and from the explicit expressions for the coefficients of the polynomial v_p above. Thus, in general the element $\tilde{u}_p(x)$ in (3.5) and in (3.7) are different, but for ease of presentation we have not explicitly written out the dependence on the parameters β_j in the elements $\tilde{u}_p$, $\{\Phi_p\}_{p=0}^{N}$ and $\alpha_{\ell,m}$.

4 A note on a fundamental sequences method for the Dirichlet problem

The proposed fundamental sequences method for the lateral Cauchy problem can be adjusted to other boundary conditions as well. Since we shall need to generate synthetic data in the numerical section via the well-posed Dirichlet problem, we outline proposed method for it. We therefore consider the following initial Dirichlet boundary value problem

$$\begin{cases} au''_{tt} + bu'_t - u''_{xx} = 0, & (x, t) \in (0, 1) \times (0, T), \\ u(0, t) = \varphi_0(t), \quad u(1, t) = \varphi_1(t), & t \in (0, T), \\ u(x, 0) = 0, \quad au'_t(x, 0) = 0, & x \in (0, 1). \end{cases} \tag{4.1}$$

Applying the Laguerre transformation or the Houbolt method give, similar to Section 3, the following sequence of boundary value problems for ordinary differential equations

$$\begin{cases} u''_p - \gamma^2 u_p = \sum_{m=0}^{p-1} \beta_{p-m} u_m, & x \in (0, 1), \\ u_p(0) = \varphi_{0,p}, \quad u_p(1) = \varphi_{1,p}, & p = 0, 1, 2, \dots . \end{cases} \tag{4.2}$$

In the case of the Laguerre transform,

$$\varphi_{\ell,p} = \int_0^\infty e^{-\kappa t} L_p(\kappa t) \varphi_\ell(t)\, dt, \quad p = 0, 1, 2, \dots, \ \ell = 0, 1,$$

whilst for the Houbolt method $\varphi_{0,p} = \varphi_0(t_p)$, $\varphi_{1,p} = \varphi_1(t_p)$, for $p = 0, 1, \dots, N$, with the same grid points as in (2.3). The coefficients β_j and γ^2 are as in Section 2.

Representing the solution to (4.2) in the form (3.3) and collocating on the endpoints of the space interval to match the Dirichlet conditions give a sequence of linear systems to find $\alpha_{1,p}$ and $\alpha_{2,p}$,

$$\begin{cases} \alpha_{1,p}\Phi_0(0, y_1) + \alpha_{2,p}\Phi_0(0, y_2) = \varphi_{0,p} - \sum_{m=0}^{p-1} \sum_{\ell=1}^{2} \alpha_{\ell,m} \Phi_{p-m}(0, y_\ell), \\ \alpha_{1,p}\Phi_0(1, y_1) + \alpha_{2,p}\Phi_0(1, y_2) = \varphi_{1,p} - \sum_{m=0}^{p-1} \sum_{\ell=1}^{2} \alpha_{\ell,m} \Phi_{p-m}(1, y_\ell), \end{cases}$$

for $p = 0, 1, \dots, N$.

As above, the numerical solution of the initial boundary value problem (4.1) in case of the Laguerre transformation is calculated as

$$u(x,t) \approx u_N(x,t) = \kappa \sum_{p=0}^{N} \tilde{u}_p(x) L_p(\kappa t) \tag{4.3}$$

and for the Houbolt method

$$u(x,t_p) \approx \tilde{u}_p(x), \quad p = 0, 1, \dots, N. \tag{4.4}$$

As explained in the final paragraph of Section 3, the element $\tilde{u}_p$ depends on the parameters β_j and thus $\tilde{u}_p$ differs in (4.3) and (4.4).

We mention that numerical methods for direct problems for the wave and telegraph equations related to the above are [11, 14, 29, 34]. Of course, for the direct problem for the wave equation (4.1), any of the standard numerical methods such as the finite element method (with or without time-discretisation), finite differences or the boundary element method can be applied although with more effort for mesh generation, see the overview [33].

5 Numerical examples

We shall present numerical examples for solving (1.1) both for the heat equation and the wave equation, as well as some results for the well-posed Dirichlet problem.

To measure the errors of the numerical solution compared with the exact solution, we use the following L_2-errors:

$$\begin{aligned} e^2(u,u_N) &= \int_0^T \int_0^1 (u(x,t) - u_N(x,t))^2 \, dxdt, \\ q^2(u,u_N) &= \frac{e^2(u,u_N)}{\int_0^T \int_0^1 u^2(x,t)\, dxdt}, \end{aligned} \tag{5.1}$$

where u is the exact solution and u_N is the numerical solution. The integrals are calculated using the trapezoid quadrature.

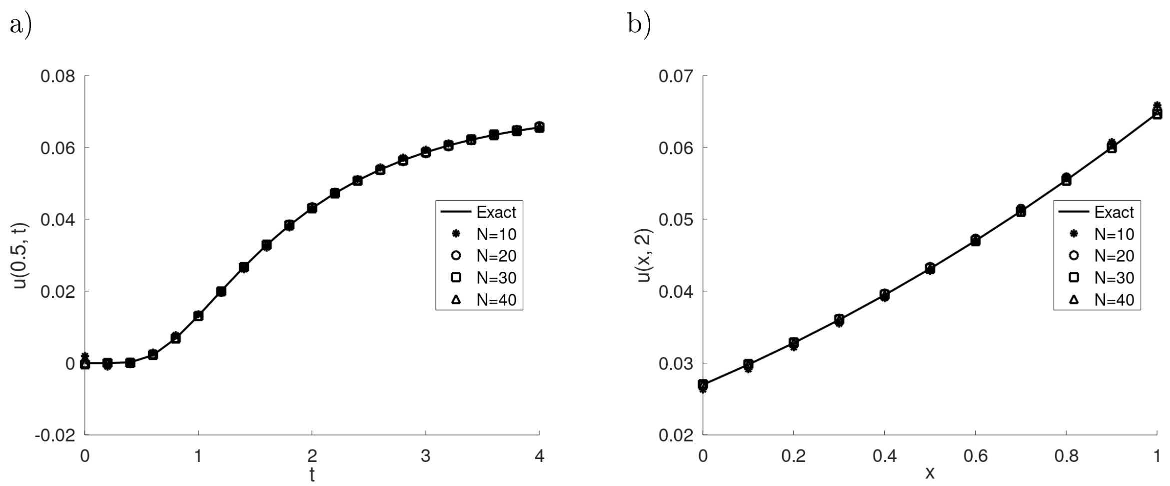

Fig. 5.1. The exact u and numerical u_N solutions at a) $(0.5,t)$ and b) $(x,2)$ for Ex. 1.1

N	$e(u, u_N)$	$q(u, u_N)$
0	$7.00e-2$	$7.19e-1$
10	$1.65e-3$	$1.70e-2$
20	$6.48e-4$	$6.65e-3$
30	$2.30e-4$	$2.37e-3$
40	$1.49e-4$	$1.53e-3$
50	$7.61e-5$	$7.82e-4$

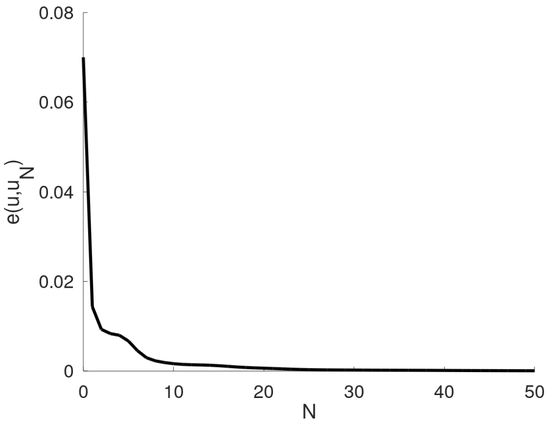

Table 5.1. L_2-errors between the exact (u) and numerical (u_N) solution for Ex. 1.1

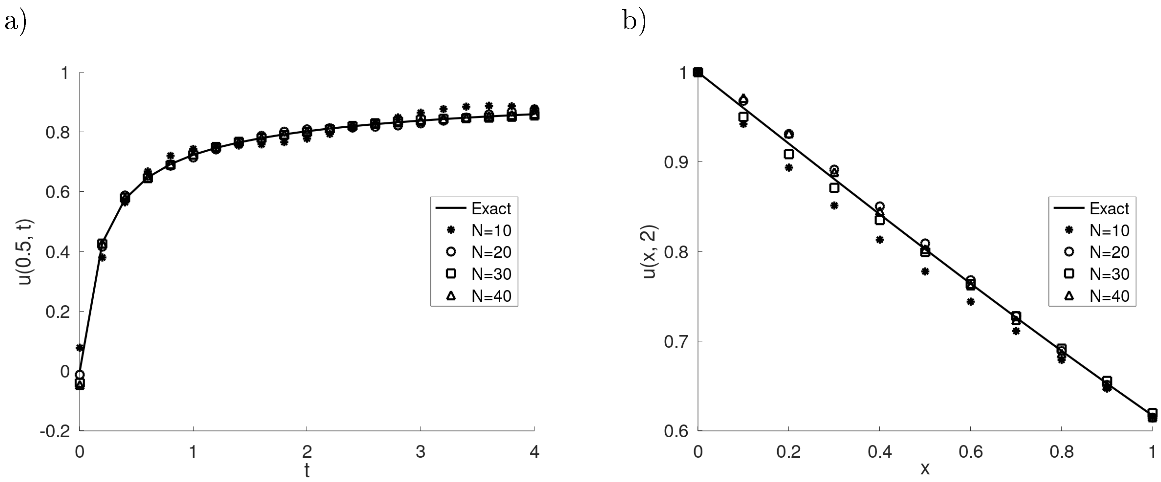

Fig. 5.2. The exact u and numerical u_N solutions at a) $(0.5, t)$ and b) $(x, 2)$ for Ex. 1.2

5.1 Example 1: Laguerre transform for the Dirichlet problem for the heat equation

In this first example, we use the Laguerre transform with scaling parameter $\kappa = 1$, for time-reduction for the heat equation with Dirichlet boundary conditions. Therefore, in (4.1) the constant $a = 0$ and $b = 1$. The two source points $y_1 < 0$ and $y_2 > 1$ should be selected not to close to the endpoints of the interval $(0, 1)$ and not too far, see [20]. Thus, we choose $y_1 = -1$ and $y_2 = 2$. As the exact solution of the problem to compare with, we consider two different functions.

1. The fundamental solution of the heat equation corresponding to a unit heat source placed at $x = 4$ and $t = 0$:

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-4)^2}{4t}}. \tag{5.2}$$

The final time is chosen as $T = 4$. Following the notation in (4.2), Laguerre transforms of the generated boundary data for $p = 0, 1, 2, \dots$ are

$$\varphi_{0,p} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\kappa t - \frac{4}{t}}}{\sqrt{t}} L_p(\kappa t)\, dt, \quad \varphi_{1,p} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\kappa t - \frac{9}{4t}}}{\sqrt{t}} L_p(\kappa t)\, dt.$$

N	$e(u, u_N)$	$q(u, u_N)$
0	$4.76e-1$	$2.85e-1$
10	$1.21e-1$	$7.23e-2$
20	$8.65e-2$	$5.18e-2$
30	$6.95e-2$	$4.16e-2$
40	$5.84e-2$	$3.50e-2$
50	$5.03e-2$	$3.01e-2$

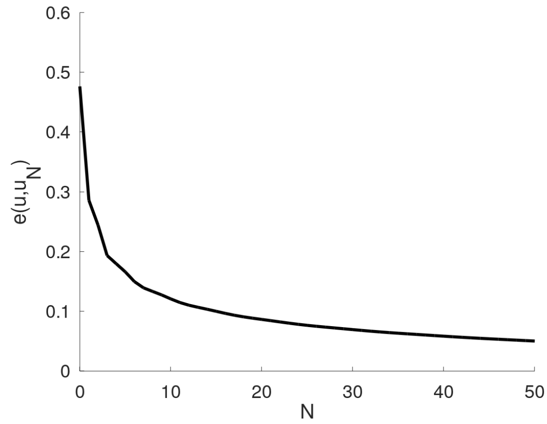

Table 5.2. L_2-errors between the exact (u) and numerical (u_N) solution for Ex. 1.2

a) b)

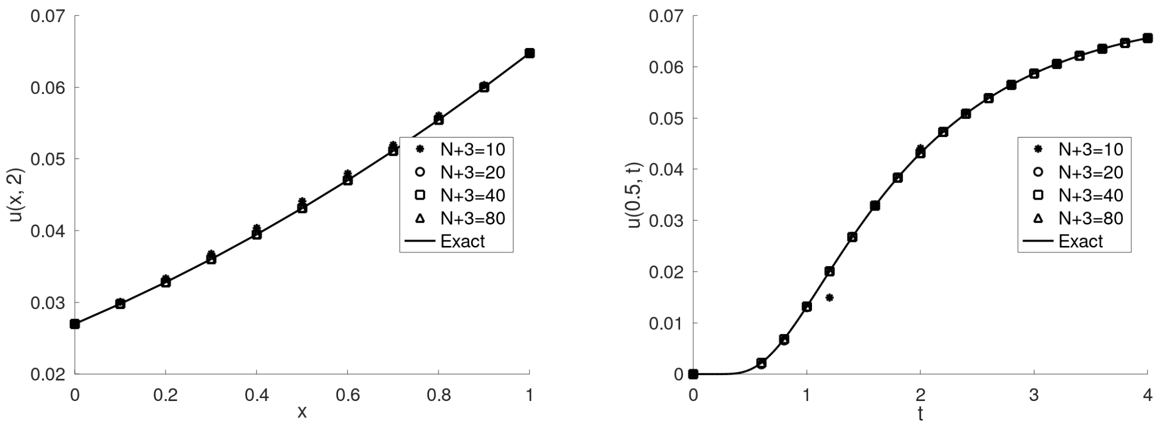

Fig. 5.3. The exact u and numerical u_N solutions at a) $(0.5, t)$ and b) $(x, 2)$ for Ex. 2.1

$N+3$	$e(u, u_N)$	$q(u, u_N)$
10	$3.53e-3$	$3.74e-2$
20	$1.97e-4$	$2.07e-3$
40	$1.73e-5$	$1.83e-4$
60	$5.93e-6$	$6.26e-5$
80	$2.58e-6$	$2.72e-5$

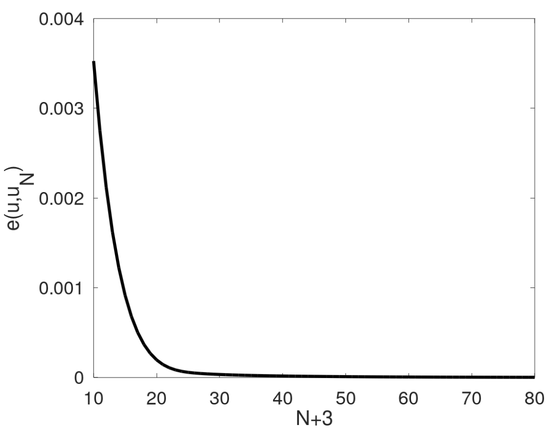

Table 5.3. L_2-errors between the exact (u) and numerical (u_N) solution for Ex. 2.1

In Fig. 5.1 are values of the exact solution $u(0.5, \cdot)$ respectively $u(\cdot, 2)$ plotted together with the corresponding numerical solution u_N in (4.3) generated by the fundamental sequences method with the Laguerre transform, outlined in Section 4, for $N = 10, 20, 30, 40$. The L_2-errors (5.1) between the exact solution u and the numerical solution u_N are presented in Table 5.1.

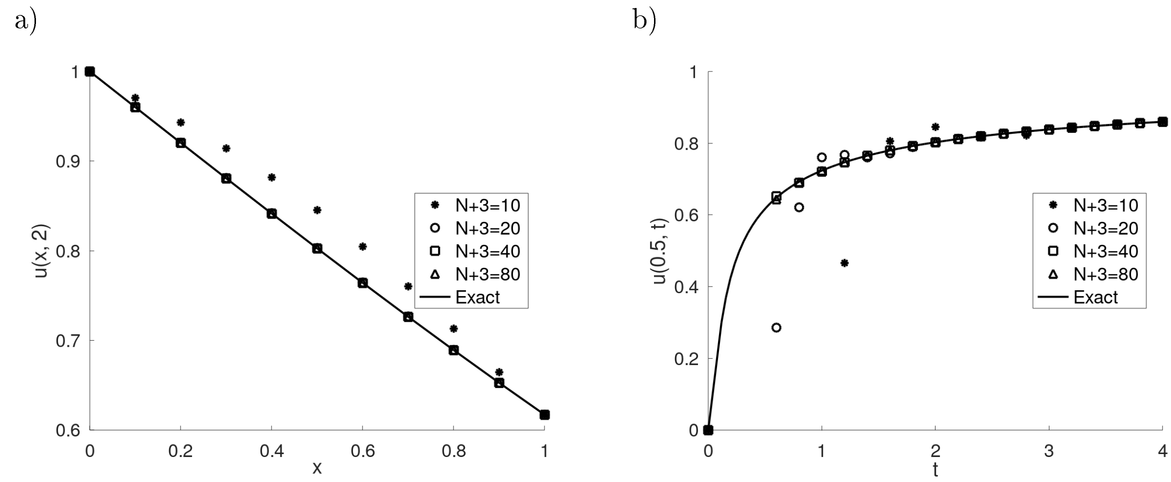

Fig. 5.4. The exact u and numerical u_N solutions at a) $(0.5, t)$ and b) $(x, 2)$ for Ex. 2.2

$N+3$	$e(u, u_N)$	$q(u, u_N)$
10	$1.88e-1$	$1.26e-1$
20	$1.17e-1$	$1.10e-1$
40	$1.35e-2$	$8.37e-2$
60	$1.10e-2$	$6.81e-2$
80	$9.47e-3$	$5.80e-2$

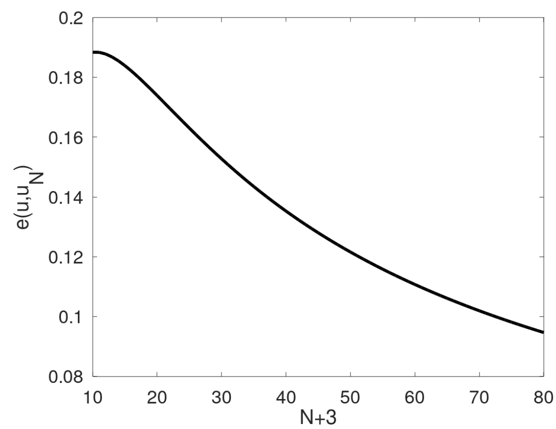

Table 5.4. L_2-errors between the exact (u) and numerical (u_N) solution for Ex. 2.2

2). As the exact solution, we now consider the function:

$$u(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{5.3}$$

It is easily checked that $u(x,t)$ satisfies the heat equation. The functions $\varphi_{\ell,p}$, for $p = 0, 1, 2, \dots$ in (4.2) are computed as

$$\varphi_{0,p} = \int_0^\infty e^{-\kappa t} L_p(\kappa t)\, dt, \quad \varphi_{1,p} = \int_0^\infty \operatorname{erfc}\left(\frac{1}{2\sqrt{t}}\right) e^{-\kappa t} L_p(\kappa t)\, dt.$$

In Fig. 5.2 are values of the exact solution $u(0.5, \cdot)$ respectively $u(\cdot, 2)$ plotted together with the numerical solutions (4.3) found by the fundamental sequences method and Laguerre transform for $N = 10, 20, 30, 40$. The L_2-errors between the exact and numerical solutions computed via (5.1) are presented in Table 5.2.

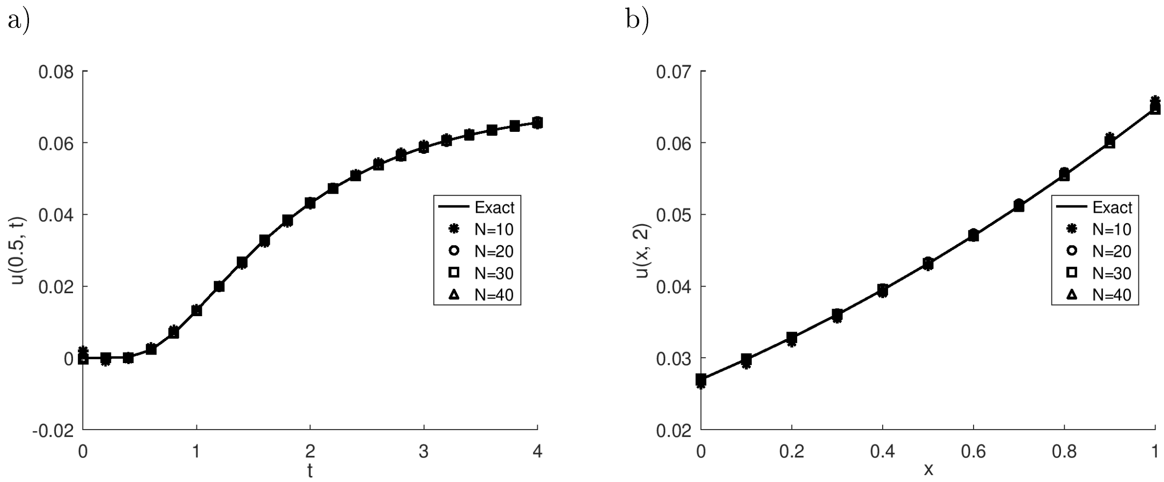

Fig. 5.5. The exact u and numerical u_N solutions at a) $(0.5, t)$ and b) $(x, 2)$ for Ex. 3

N	$e(u, u_N)$	$q(u, u_N)$
0	$9.01e-2$	$9.35e-1$
10	$4.55e-3$	$4.68e-2$
20	$1.06e-3$	$1.09e-2$
30	$2.89e-4$	$2.97e-3$
40	$1.82e-4$	$1.86e-3$
50	$1.00e-4$	$1.03e-3$

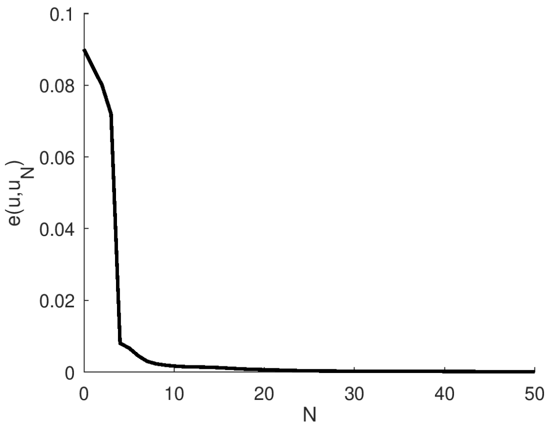

Table 5.5. L_2-errors between the exact (u) and numerical (u_N) solution for Ex. 3

5.2 Example 2: Houbolt method for the Dirichlet problem for the heat equation

To see whether it matters which time-reduction one uses, we consider the same two configuration as in Ex. 1 but apply instead the Houbolt method for the heat equation. Thus, as in Ex. 1, $a = 0$ and $b = 1$. The source points for the fundamental sequences method are $y_1 = -1$ and $y_2 = 2$. As the exact solution of the problem, we consider the two functions of Ex. 1.

1). The function (5.2) is used to generate data. In Fig. 5.3 are values of the exact solution $u(0.5, \cdot)$ respectively $u(\cdot, 2)$ plotted together with approximations (4.4) for $N + 3 = 10, 20, 40, 80$. The L_2-errors computed by (5.1) are presented in Table 5.3.

2). The error function (5.3) is used to generate data. In Fig. 5.4 are values of the exact solution $u(0.5, \cdot)$ respectively $u(\cdot, 2)$ plotted together with the numerical solutions (4.3) found by the fundamental sequences method and the Houbolt method for $N+3 = 10, 20, 40, 80$. The L_2-errors computed by (5.1) are presented in Table 5.4.

Compared with the results of Ex. 1 there are no dramatic changes, although comparing Fig. 5.2 with Fig. 5.4 and the results of Table 5.2 and Table 5.4, the Houbolt method is slightly less accurate when using the exact solution (5.3). Thus, in the experiments for the lateral Cauchy problem (1.1),

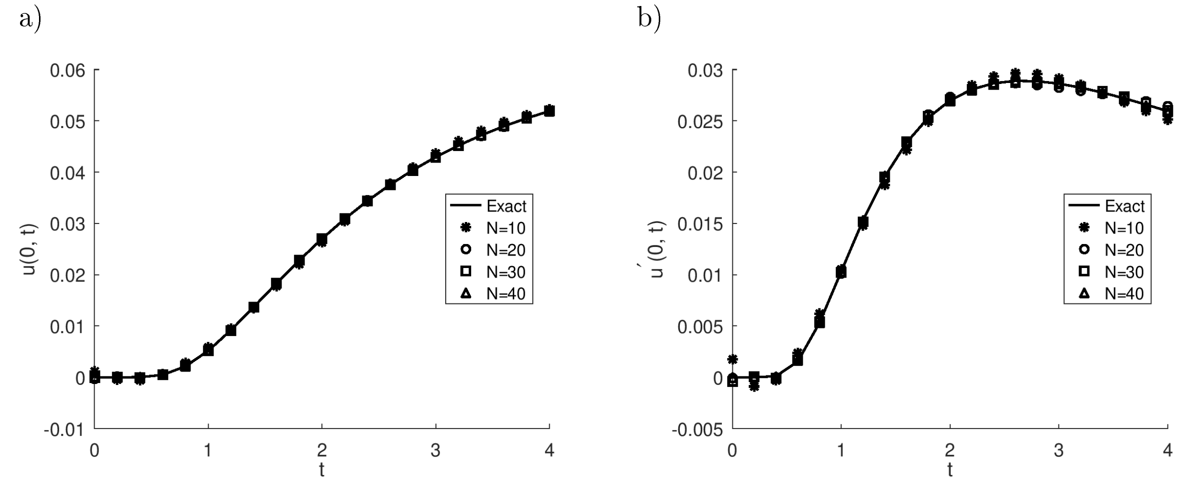

Fig. 5.6. The reconstructed Cauchy data: a) function values and b) derivative at $(0, t)$ for Ex. 3

N	$e(u, u_N)$	$q(u, u_N)$
0	$7.01e-2$	$7.20e-1$
10	$1.74e-3$	$1.79e-2$
20	$7.11e-4$	$7.30e-3$
30	$4.52e-4$	$4.64e-3$
40	$4.68e-4$	$4.81e-3$
50	$4.65e-4$	$4.77e-3$

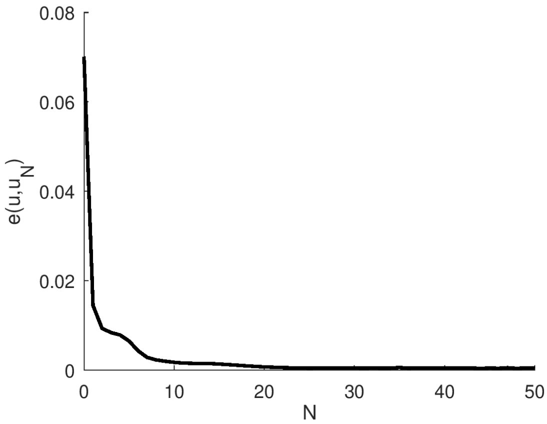

Table 5.6. L_2-errors between the exact (u) and numerical (u_N) solution with 3% noise for Ex. 3

we feel no need to present results for both methods but focus on the fundamental sequences method with the Laguerre transform.

5.3 Example 3: Laguerre transform for the lateral Cauchy problem for the heat equation

We consider the lateral Cauchy problem (1.1) for the heat equation, thus $a = 0$ and $b = 1$. The Laguerre transform is applied for the time-reduction. The source points for the fundamental sequences method are selected as $y_1 = -1$ and $y_2 = 2$. As an exact solution we use (5.2).

In Fig. 5.5 are the values of the exact solution $u(0.5, \cdot)$ respectively $u(\cdot, 2)$ plotted together with the numerical solution (3.5) found by the fundamental sequences method with the Laguerre transform for $N = 10, 20, 30, 40$. The L_2-errors computed by (5.1) are presented in Table 5.5. The reconstructed Cauchy data when $x = 0$ are presented in Fig. 5.6. It is particularly pleasing to see that also the derivative is accurately reconstructed. This is partly due to the explicit formula (3.6) for the derivative of the numerical solution (3.5).

The results of the similar numerical experiments, but with noisy data (noise level $\delta = 3\%$) are presented in Figs. 5.7–5.8 and Table 5.6. Results with noisy data are of course less accurate but importantly noise is not magnified out of proportion.

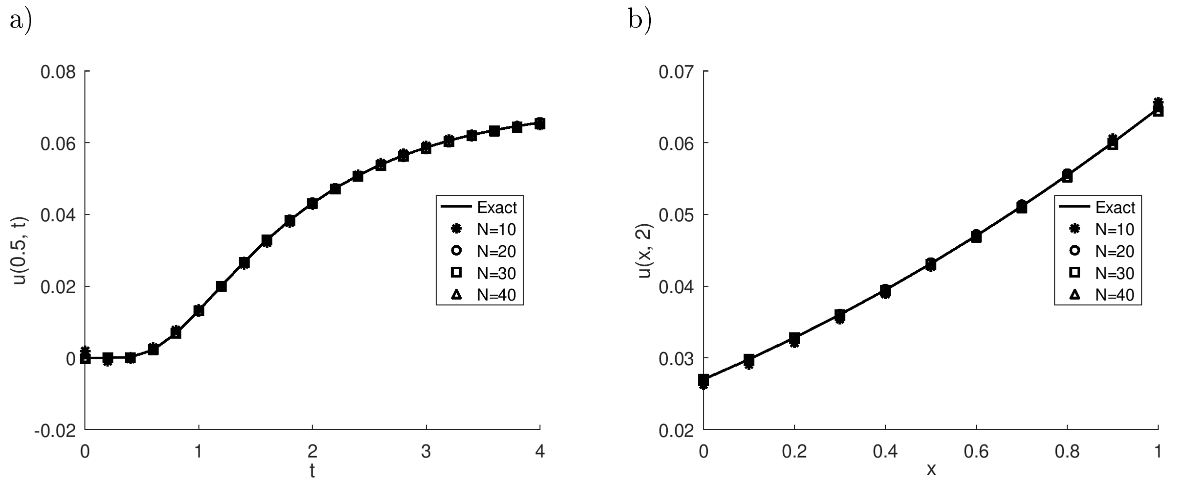

Fig. 5.7. The exact u and numerical u_N solutions at a) $(0.5, t)$ and b) $(x, 2)$ with 3% noise for Ex. 3

N	$\tilde{e}(u, u_N)$	$\tilde{e}(u', u'_N)$
0	$5.24e-2$	$9.14e-2$
10	$1.56e-3$	$2.49e-2$
20	$5.64e-6$	$2.05e-2$
30	$2.92e-6$	$1.91e-2$
40	$1.92e-6$	$1.59e-2$
50	$1.10e-6$	$1.13e-2$

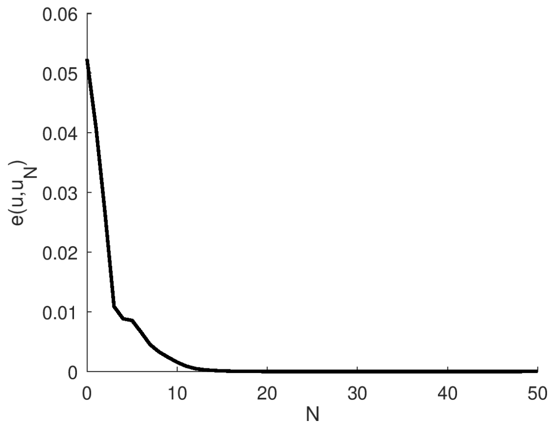

Table 5.7. L_2-errors of the numerical solution u_N and derivative u'_N, computed by (5.4), for Ex. 4

The regularization parameter for the Tikhonov regularization is in the case of exact data $1e-10$ and for noisy data $1e-3$.

5.4 Example 4: Laguerre transform for the Cauchy problem for the wave equation

We consider the lateral Cauchy problem (1.1) for the wave equation, thus $a = 1$ and $b = 0$. The Laguerre transform is applied for the time-reduction with scaling parameter $\kappa = 1$. The source points for the fundamental sequences method are $y_1 = -1$ and $y_2 = 2$. In this example, the exact solution is unknown and we instead use synthetic data. The lateral Cauchy data is generated by numerically solving the Dirichlet problem (4.1), i.e., constructing (4.3), with the boundary functions chosen as

$$\varphi_0(t) = \varphi(0,t),\ \varphi_1(t) = \varphi(1,t),\ \ \varphi(x,t) = \frac{1}{10\pi} t^2 e^{-t+2} \cos(\pi x + 1), t \in (0, 4].$$

The lateral Cauchy data is then φ_0 and $\tilde{\varphi}_1(t)$ with $\tilde{\varphi}_1(t)$ obtained as the derivative of (4.3) when $x = 1$ (similar to (3.6)). Note that for the lateral Cauchy problem for the wave equation, uniqueness of a solution only holds in a space-time interval with time a function of space [3,4,28]. However, the

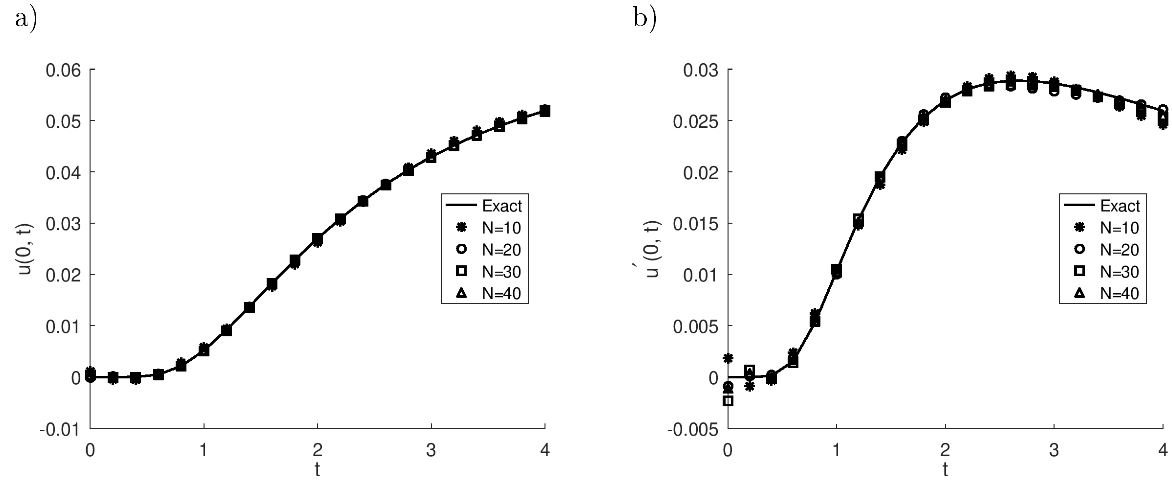

Fig. 5.8. The reconstructed Cauchy data: a) function values and b) derivative at $(0,t)$ with 3% noise for Ex. 3

N	$\tilde{e}(u,u_N)$	$\tilde{e}(u',u'_N)$
0	$5.24e-2$	$9.14e-2$
5	$8.57e-3$	$2.88e-2$
10	$1.56e-3$	$2.43e-2$
15	$1.54e-4$	$2.09e-2$
20	$1.39e-4$	$1.74e-2$
25	$2.51e-3$	$2.55e-2$

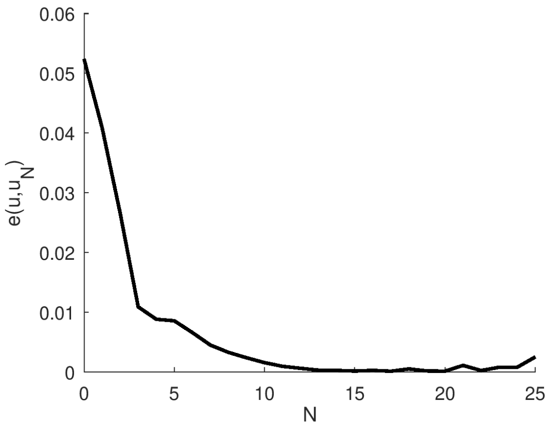

Table 5.8. L_2 errors of the numerical solution u_N and derivative u'_N, computed by (5.4), with 3% noise for Ex. 4

chosen data above can be extended for all $t>0$, thus according to [22] uniqueness holds throughout the time interval without dependence on space.

The Laguerre transform of φ_0 and φ_1 are

$$\varphi_{0,p}=\varphi_p(0),\ \varphi_{1,p}=\varphi_p(1),\ \ \varphi_p(x)=\frac{e^2(2+\kappa p(\kappa(p-1)-4))}{10\pi(\kappa+1)^{p+3}}\cos(\pi x+1).$$

The data for the time-transformed Cauchy problems for the wave equation (2.2) consists of $\varphi_{1,p}$ and the numerically generated derivative $\tilde{\varphi}_{1,p}=\tilde{u}'_p(1)$ (with $\tilde{u}_p$ as in the sum in the right-hand side of the approximation (4.3) for the Dirichlet problem).

In this example, we investigate the obtained lateral Cauchy data when $x=0$ and $0<t<4$. As the exact lateral Cauchy data we use $\varphi_1(t)$ and the generated element $\tilde{\varphi}_1(t)$ numerically calculated from the Dirichlet problem for the wave equation, as explained above, for $N=60$. Furthermore, as exact Cauchy data to compare against when $x=0$, we take φ_0 and $\tilde{\varphi}_0(t)$ with $\tilde{\varphi}_0(t)$ obtained as the derivative of the numerical solution (4.3) of the Dirichlet problem when $x=1$ (similar to the construction of $\tilde{\varphi}_1(t)$).

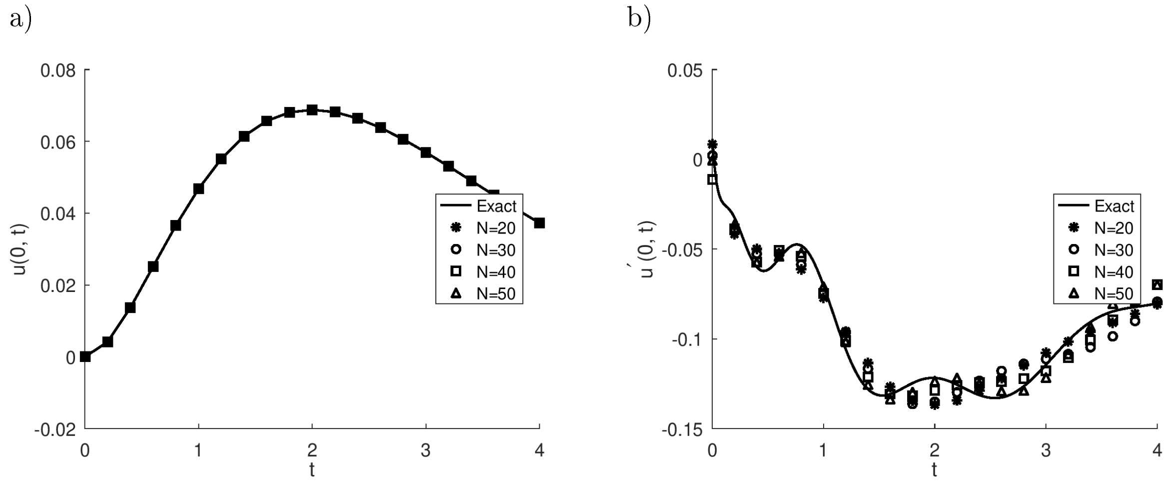

Fig. 5.9. The reconstructed Cauchy data: a) function values and b) derivative at $(0,t)$ for Ex. 4

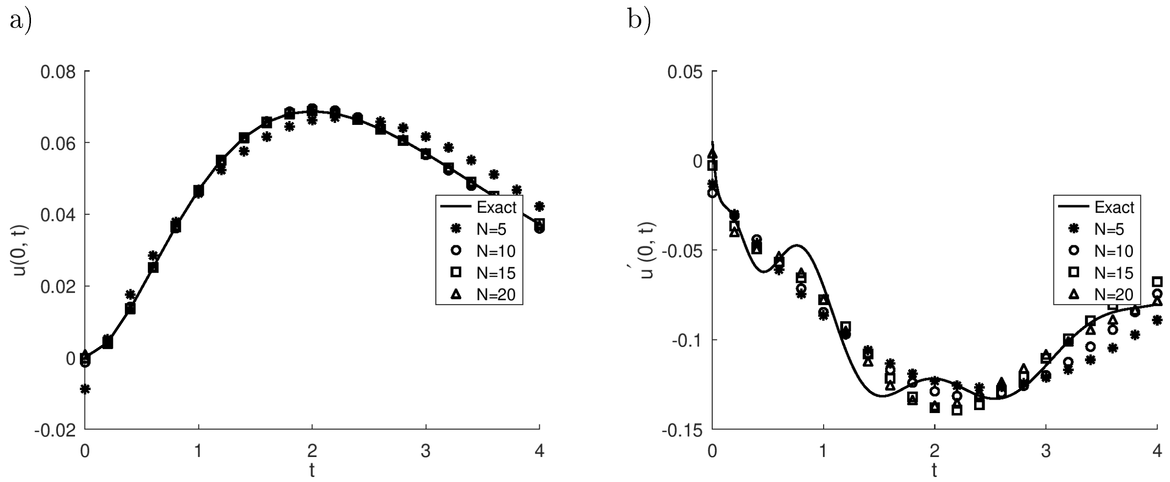

Fig. 5.10. The reconstructed Cauchy data: a) function values and b) derivative at $(0,t)$ with 3% noise for Ex. 4

We consider the L_2-error:

$$\tilde{e}^2(u,u_N) = \int_0^T (u(0,t) - u_N(0,t))^2 \, dt. \tag{5.4}$$

The L_2-errors of the reconstruction of the lateral Cauchy data (u_N and u'_N from (3.5) and (3.6), respectively) when $x = 0$ and $0 < t < 4$, computed by (5.4) are presented in Table 5.7. The reconstructed Cauchy data when $x = 0$ and $0 < t < 4$ for different values of the parameter N together with the exact Cauchy data are presented in Fig. 5.9. The regularization parameter is chosen as $1E-11$. Since the given data is not exact it is expected to see an error in the reconstructions in particular for the derivative. Still, the reconstructions follow the (numerically) exact solutions.

The results of the similar numerical experiments, but with noisy data (noise level $\delta = 3\%$) are given in Table 5.8 and in Fig. 5.10. In the case of noisy data, the results are presented only until a threshold value $N = 20$; for larger values of N, the L_2-error starts to increase. The regularization parameter is chosen as $1E-5$. We now also see a larger error in the functions values. The reconstructions still captures the main features both for the function values and the derivatives.

Thus, the generated approximations are stable with respect to (moderate) noise.

Let us mention that the presented numerical results are obtained using Octave and performed on an ordinary workstation having an Intel(R) Core(TM) i7 CPU at 2.60 GHz. To give an example of the computational time, in Ex. 1 (Dirichlet problem for the heat equation) with $N = 50$ in (4.3) the computations take about 0.15 seconds.

6 Conclusion

A fundamental sequences method has been derived for the one-dimensional lateral Cauchy problem for the damped wave equation. Either the Laguerre transform or the Houbolt scheme is applied for time-reduction rendering a sequence of second-order ordinary differential equations (ODEs) with data given at the right endpoint of a finite space interval. The study includes as a special case the classical lateral Cauchy problem for the heat equation. A fundamental sequence to the ODEs is explicitly constructed and the solution to the ODEs are expressed as a linear combination of elements of this fundamental sequence. Collocating at the right endpoint of the space interval to match the transformed Cauchy data renders linear systems for finding the coefficients in the expansion. Tikhonov regularization is applied for the stable solution of the linear systems. A similar fundamental sequences method for the well-posed Dirichlet boundary value problem for the damped wave equation is included as well. Numerical examples for both the Dirichlet and lateral Cauchy problem for the heat and wave equations, using the Laguerre transform as well as the Houbolt method, with exact and noisy data are given showing the efficiency and accuracy of the proposed method.

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