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ON THE NUMERICAL INTEGRATION OF SINGULAR DOUBLE INTEGRALS USING GREEN'S THEOREM

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Резюме. Розглянуто наближене обчислення подвійних інтегралів з особливостями за допомогою теореми Гріна, де точка спостереження належить області або лежить на межі області. Використовуючи відповідну формулу, подвійний інтеграл по однозвязній області зведено до криволінійного інтеграла. Після параметризації наближене значення обчислено за допомогою складеної квадратурної формули трапецій для періодичних функцій. Для порівняння приведено метод радіального інтегрування, а також підхід з використанням кубатурних формул до подвійних інтегралів після відповідної заміни змінних. Наведено чисельні експерименти, які відображають характеристики та ефективність застосування кожного з трьох методів.

Abstract. We considered an approximate calculation of singular double integrals based on Green's theorem, where a source point belongs to a domain or it is lying on a boundary of the domain. Using the corresponding formula a double integral over a simply connected domain is reduced to a boundary integral. After parameterization and applying the composite trapezoidal rule for periodic functions the approximate value of the integral is calculated. For the comparison, we provided also the radial integration method and the approach of double integrals calculation using cubatures after an appropriate change of variables. The numerical results that represent the effectiveness of each of these three methods are given at the end.

1. Introduction

A lot of problems of mathematical physics that are defined in a domain and described by differential equations can be reduced to a boundary integral equation. It means that the dimension of the problem is decreased and, as a consequence, it usually decreases the number of computational efforts needed to solve the problem. However, often (for instance, in a case when the fundamental solution of a differential equation is unknown, in general) such types of equations can be reduced only to a boundary-domain integral equation (BDIE) where the necessity of the numerical integration of domain integrals remains.

For two-dimensional domains, we obtain double integrals. There can be considered two types of double integrals: double integrals with known integrand and double integrals with unknown integrand. An example of such integrals is provided in [5], where the left side of the system of BDIEs contains the first-type integrals and the right side includes double integral with known integrand.

Key words. Singular double integrals; Green's theorem; Radial integration method; Quadratures.

Knowing that type it is possible to choose the most efficient and expedient approach to calculate an integral.

For the elliptic equation with variable coefficients that considered in [5] and based on the operator A such that

$$Au = \nabla \cdot (\sigma \nabla u)$$

with known function $\sigma > 0$, a parametrix function P was used to obtain BDIEs. The parametrix should satisfy the following expression (see [13])

$$A_x P(x, y) = \delta(x - y) + R(x, y),$$

where δ is the Dirac function and the remainder function R has a weak singularity for $x = y$. The parametrix can be chosen as

$$P(x, y) = \frac{\ln |x - y|}{2\pi\sigma(y)}, \quad x, y \in \mathbb{R}^2$$

and then the corresponding remainder function is

$$R(x, y) = \frac{(x - y) \cdot \nabla\sigma(x)}{2\pi\sigma(y)|x - y|^2}, \quad x, y \in \mathbb{R}^2.$$

We consider the double integrals based on these two functions, but with some simplifications to observe the behavior and order of approximations for the respective singular integrals

$$P_s(x, y) = \ln |x - y|, \quad x, y \in \mathbb{R}^2,$$

$$R_s(x, y) = \frac{(x - y) \cdot \nabla\sigma(x)}{|x - y|^2}, \quad x, y \in \mathbb{R}^2.$$

Let D be a simply connected bounded domain in $\mathbb{R}^2$ with boundary $\partial D \in C^2$ and $\sigma \in C^1(\overline{D})$, $\sigma > 0$. Let us denote $\Gamma = \partial D$. The aim is to calcualte the following integrals

$$I_1(x) = \frac{1}{2\pi} \int\limits_D \frac{\partial P_s(x, y)}{\partial \nu(x)} dy = \frac{1}{2\pi} \int\limits_D \frac{(x - y) \cdot \nu(x)}{|x - y|^2} dy, \quad x \in \Gamma, \tag{1}$$

$$I_2(x) = \frac{1}{2\pi} \int\limits_D R_s(x, y) dy = \frac{1}{2\pi} \int\limits_D \frac{(x - y) \cdot \nabla\sigma(x)}{|x - y|^2} dy, \quad x \in D, \tag{2}$$

$$I_3(x) = \frac{1}{2\pi} \int\limits_D P_s(x, y) dy = \frac{1}{2\pi} \int\limits_D \ln |x - y| dy, \quad x \in D, \tag{3}$$

where $\nu = (\nu_1, \nu_2)$ – outward unit normal vector to the boundary Γ.

The motivation of considering such integrals is that they appear in BDIEs (see, for example, [3,4]), which are an equivalent form of boundary value problems for the elliptic equation with variable coefficients defined by the operator A.

The first approach that will be used to calculate integrals defined above is based on Green's theorem and corresponding formula. The second one – the radial integration method (RIM) allows having a deal with boundary integrals instead of double integrals. Finally, the last method that will be considered is an application of quadrature formulas directly to double integrals after some change of variables (see [3]). The goal of this article is to compare the effectiveness of each of these methods that applied to singular double integrals (1)-(3) with known integrands.

For the outline of the work, in Section 2, we reduce double integrals to boundary integrals using Green's theorem, rewrite them as ordinary integrals over $[0, 2\pi]$ taking into account boundary parameterization and splitting singularities from some kernels where it is needed. A brief schema of RIM application and the approach based on quadratures for double integrals are presented in Section 3. Numerical integration of integrals for all three methods is provided in Section 4. In Section 5, a few numerical examples for different domains and functions σ are considered. The main remarks and conclusions are given in Section 6.

2. Green's theorem application

For the integral I_1 the source point x will not coincide with the field point y, although these points can be close to each other as much as possible. It is so-called a nearly singular integral.Since R_s as a simplification of the remainder function R has the same weak singularity when $x = y$, I_2 is integrable. The last one integral I_3 has a logarithmic singularity.

Let us recall Green's theorem that shows the connection between a line integral around a closed curve Γ and a double integral over the region bounded by that curve. Let Γ be a positively oriented, piecewise smooth, a Jordan curve in a plane, and let D is the region bounded by Γ. If L, Q are functions of (y_1, y_2) defined on an open region containing D and having continuous partial derivatives there, then

$$\iint\limits_D \Big(\frac{\partial L}{\partial y_1} - \frac{\partial Q}{\partial y_2}\Big) dy_1 dy_2 = \int\limits_\Gamma Q dy_1 + L dy_2 \tag{4}$$

where the path of integration along Γ is anticlockwise [6]. We assume that the boundary Γ has the following parametric representation

$$\Gamma = \{x(t) = (x_1(t), x_2(t)),\ t \in [0, 2\pi]\}. \tag{5}$$

Since $y = (y_1, y_2) = (x_1(\tau), x_2(\tau))$ we can rewrite the right side of the (4) in the following form

$$\int\limits_\Gamma Q dy_1 + L dy_2 = \int\limits_0^{2\pi} \Big[Q(x_1(\tau), x_2(\tau))x_1'(\tau) + L(x_1(\tau), x_2(\tau))x_2'(\tau)\Big] d\tau. \tag{6}$$

Based on formulas (4), (6) and knowing functions L and Q it is possible to reduce double integrals (1)-(3) to ordinary integrals over $[0, 2\pi]$ to apply corresponding quadratures.

Let's consider $I_1(x)$ and provide transformation steps to obtain functions L and Q. Hence,

$$\frac{(x_1-y_1)\nu_1(x)+(x_2-y_2)\nu_2(x)}{|x-y|^2} \Rightarrow$$
$$\frac{\partial L}{\partial y_1}=\frac{(x_1-y_1)\nu_1(x)}{|x-y|^2};\ \frac{\partial Q}{\partial y_2}=\frac{-(x_2-y_2)\nu_2(x)}{|x-y|^2}.$$

Therefore

$$L=-\ln|x-y|\nu_1(x),\quad Q=\ln|x-y|\nu_2(x).$$

Here and further we omit variables of functions L, Q and assume that they are the functions of points x and y. Substituting this into (1) and taking into account (5) we obtain that

$$I_1(x(t))=\frac{1}{2\pi}\int_0^{2\pi}\ln|x(t)-x(\tau)|\Big(\nu_2(x(t))x_1'(\tau)-\nu_1(x(t))x_2'(\tau)\Big)d\tau, \qquad t\in[0,2\pi]. \tag{7}$$

We denote the integrand in (7) as a function $H_1(t,\tau)$. It is easy to see that $H_1(t,\tau)$ has a logarithmic singularity and following the approach from the [11] it can be rewritten as

$$H_1(t,\tau)=H_1^{(1)}(t,\tau)\ln\frac{4}{e}\sin^2\Big(\frac{t-\tau}{2}\Big)+H_1^{(2)}(t,\tau)$$

with

$$H_1^{(1)}(t,\tau)=\frac{1}{2}\Big(\nu_2(x(t))x_1'(\tau)-\nu_1(x(t))x_2'(\tau)\Big)$$

and

$$H_1^{(2)}(t,\tau)=\begin{cases}\dfrac{1}{2}\Big(\nu_2(x(t))x_1'(\tau)-\nu_1(x(t))x_2'(\tau)\Big)\ln\dfrac{|x(t)-x(\tau)|^2}{\frac{4}{e}\sin^2(\frac{t-\tau}{2})} & \text{for}\quad t\neq\tau,\\[2ex] -\dfrac{1}{2}|x'(t)|\ln\big(e|x'(t)|^2\big) & \text{for}\quad t=\tau.\end{cases}$$

Considering $I_2(x)$ and making the similar steps as for $I_1(x)$ we obtain

$$I_2(x)=\frac{1}{2\pi}\int_0^{2\pi}H_2(x,\tau)d\tau,\quad x\in D, \tag{8}$$

where

$$H_2(x,\tau)=\ln|x-x(\tau)|\Big(\frac{\partial\sigma(x)}{\partial x_2}x_1'(\tau)-\frac{\partial\sigma(x)}{\partial x_1}x_2'(\tau)\Big).$$

In this case, we avoid singularity since x belongs to the domain D and integration is over the curve Γ.

Finally, transformations for I_3

$$\ln|x-y|=0.25\ln|x-y|^2-(-0.25\ln|x-y|^2)=\frac{\partial L}{\partial y_1}-\frac{\partial Q}{\partial y_2}.$$

After integration of partial derivatives of L and Q we obtain that

$$L = \frac{1}{4}\Big[(y_1 - x_1)\ln|x-y|^2 + 2(x_2 - y_2)\arctan\Big(\frac{x_1 - y_1}{y_2 - x_2}\Big) - 2y_1\Big],$$

$$Q = -\frac{1}{4}\Big[(y_2 - x_2)\ln|x-y|^2 + 2(x_1 - y_1)\arctan\Big(\frac{y_2 - x_2}{x_1 - y_1}\Big) - 2y_2\Big].$$

Grouping those terms in L and Q and due to the boundary representation (5) the integral $I_3(x)$ can be written as

$$I_3(x) = \frac{1}{2\pi}\int\limits_0^{2\pi}\Big[H_3^{(1)}(x,\tau) + H_3^{(2)}(x,\tau) + H_3^{(3)}(x,\tau)\Big]d\tau, \quad x \in D \tag{9}$$

with

$$H_3^{(1)}(x,\tau) = \ln|x - x(\tau)|\Big((x_2 - x_2(\tau))x_1'(\tau) + (x_1(\tau) - x_1)x_2'(\tau)\Big),$$

$$H_3^{(2)}(x,\tau) = \frac{1}{2}\Big[(x_1(\tau) - x_1)\arctan\Big(\frac{x_2(\tau) - x_2}{x_1 - x_1(\tau)}\Big) +$$

$$+(x_2 - x_2(\tau))\arctan\Big(\frac{x_1 - x_1(\tau)}{x_2(\tau) - x_2}\Big)\Big],$$

$$H_3^{(3)}(x,\tau) = \frac{1}{2}\Big[x_2(\tau)x_1'(\tau) - x_1(\tau)x_2'(\tau)\Big].$$

Note, that $H_3^{(2)}$ is bounded and in case we have zero in any denominator of these two fractions the limit will be zero for that specific term.

3. The RIM and the quadratures-based approach

Double integrals mentioned in the introduction also can be calculated by the RIM that has been developed by Gao [7,8]. Following the specific steps, this method allows to compute boundary integrals instead of double integrals and avoid singularities. As advantages, we can point out that steps are universal for different integrals and are the same for two and three-dimensional domains. However, in the case of unknown integrand there should be some approximation (for instance, by radial basis functions) to use this method. Also, not always the integral over $[0,1]$ that appears in those steps can be calculated analytically, so some numerical methods should be applied as well. Numerical evaluation of arbitrary singular domain integrals using the RIM can be found in [9,10].

Following the RIM steps from [1,2] a double integral over two-dimensional domain D that bounded by a boundary Γ with known integrand $f(y)$ with a field point $y = (y_1, y_2)$ and the source point $x = (x_1, x_2)$, can be transformed into a boundary integral as described below.

1. The main expression is

$$\int\limits_D f(y)dy = \int\limits_\Gamma \frac{1}{r^d}\frac{\partial r}{\partial \nu}F(y)ds(y), \tag{10}$$

where

$$F(y) = \int_0^{r(y)} f(y) r^d dr \tag{11}$$

and $d = 1$ for two or $d = 2$ for three-dimentional space, respectively.

2. In order to evaluate the radial integral in (11), the coordinates y_1, y_2 have to be presented in terms of the distance r using

$$y_i = x_i + r_{,i} r, \quad i = 1, 2$$

where x_i and $r_{,i}$ are constant for the integral in (11), with $r_{,i} = \frac{y_i - x_i}{r}$.

3. Introducing the change of variable

$$r = z|y - x|, \quad z \in [0, 1]$$

and substituting it in the straight-line radial integral in (11), we obtain

$$F(y) = \int_0^1 f(x_1 + r_{,1} rz, x_2 + r_{,2} rz) r^2 z dz. \tag{12}$$

Note, that the integral in (12) usually can be calculated analytically, but in some cases, numerical integration is still required. So, at first, we need to calculate (12) considering all changes of variables and then calculate (10). Taking this into consideration the integrals (1)-(3) can be represented as follows

$$\begin{aligned} I_1(x(t)) =& \frac{1}{2\pi} \int_\Gamma \frac{1}{r} \frac{\partial r}{\partial \nu} (-r_1 \nu_1(x) - r_2 \nu_2(x)) ds(y) = \\ =& \frac{1}{2\pi} \int_0^{2\pi} S_1(t, \tau) d\tau, \ t \in [0, 2\pi], \end{aligned} \tag{13}$$

where

$$S_1(t, \tau) = \begin{cases} \dfrac{(x(\tau) - x(t)) \cdot \nu(x(\tau))((x(t) - x(\tau)) \cdot \nu(x(t)))}{|x(\tau) - x(t)|^2} |x'(\tau)| \\ \qquad \text{for} \quad t \neq \tau, \\ 0 \qquad \text{for} \quad t = \tau; \end{cases}$$

$$\begin{aligned} I_2(x) =& \frac{1}{2\pi} \int_\Gamma \frac{1}{r} \frac{\partial r}{\partial \nu} \Big(- r_1 \frac{\partial \sigma}{\partial x_1} - r_2 \frac{\partial \sigma}{\partial x_2} \Big) ds(y) = \\ =& \frac{1}{2\pi} \int_0^{2\pi} S_2(x, \tau) d\tau, \quad x \in D, \end{aligned} \tag{14}$$

where

$$S_2(x, \tau) = \frac{(x(\tau) - x) \cdot \nu(x(\tau))((x - x(\tau)) \cdot \nabla \sigma(x))}{|x(\tau) - x|^2} |x'(\tau)|;$$

$$I_3(x) = \frac{1}{2\pi} \int\limits_{\Gamma} \frac{1}{r} \frac{\partial r}{\partial \nu} \Big(\frac{1}{4} (\ln r^2 - 1) \Big) ds(y) = \frac{1}{2\pi} \int\limits_0^{2\pi} S_3(x, \tau) d\tau, \quad x \in D \tag{15}$$

with

$$S_3(x, \tau) = (x(\tau) - x) \cdot \nu(x(\tau))(0.25(\ln(|x(\tau) - x|^2) - 1))|x'(\tau)|.$$

At last, let us recall the calculation of double integrals I_1, I_2 and I_3 as a simplification of integrals from the papers [3, 4] using the appropriate change of variables with further application of quadratures.

We assume that Γ is star-shaped curve, so there is a one-to-one mapping defined below

$$p(\eta, t) = (\eta x_1(t), \eta x_2(t)) : \Pi \to D,$$

where $\Pi = \{(0, 0) \cup (0, 1) \times [0, 2\pi]\}$ and Jacobian $J(\eta, t) = \eta(x_1(t)x_2'(t) - x_2(t)x_1'(t))$. Therefore it is possible to parametrize the domain integrals via change of variables $y = p(\xi, \tau)$ and $x = p(\eta, t)$ with further integration over Π.

For I_1 we have the following integrand

$$\widehat{P}_s(t; \xi, \tau) = \frac{\partial P_s(x(t), \xi x(\tau))}{\partial \nu(x(t))} J(\xi, \tau).$$

For I_2 we have

$$\widetilde{R}_s(\eta, t; \xi, \tau) = R_s(p(\eta, t), p(\xi, \tau))J(\xi, \tau).$$

The strong singularity in $\widetilde{R}$ can be handled by applying the ideas from [12]. Using that approach, we can represent $\widetilde{R}_s(\eta, t; \eta, \tau)$ (see more details in [3]) in the following form

$$\widetilde{R}_s(\eta, t; \eta, \tau) = \widetilde{R}_s^{(1)}(\eta, t; \eta, \tau) + \widetilde{R}_s^{(2)}(\eta, t; \eta, \tau) \cot \frac{\tau - t}{2}$$

with

$$\widetilde{R}_s^{(1)}(\eta, t; \eta, \tau) = \frac{1}{\eta} (\nabla \sigma(\eta x(t)) \cdot \nu(t)) K_1(t, \tau) J(\eta, \tau) - \frac{1}{\eta |x'(t)|} (\nabla \sigma(\eta x(t)) \cdot \theta(t)) K_2(t, \tau) J(\eta, \tau)$$

and

$$\widetilde{R}_s^{(2)}(\eta, t; \eta, \tau) = -\frac{1}{2\eta |x'(t)|} (\nabla \sigma(\eta x(t)) \cdot \theta(t)) J(\eta, \tau),$$

where functions K_1, K_2 are defined in [3].

Finally, for I_3 we have

$$\widetilde{P}_s(\eta, t; \xi, \tau) = P_s(p(\eta, t), p(\xi, \tau))J(\xi, \tau),$$

The function $\widetilde{P}_s(\eta, t; \xi, \tau)$ has a logarithmic singularity when $\eta = \xi$. Therefore, $\widetilde{P}_s(\eta, t; \eta, \tau)$ can be rewritten (see [11]) as follows

$$\widetilde{P}_s(\eta, t; \eta, \tau) = \widetilde{P}_s^{(1)}(\eta, \tau) \ln\Big(\frac{4}{e} \sin^2 \frac{t - \tau}{2} \Big) + \widetilde{P}_s^{(2)}(\eta, t; \eta, \tau),$$

where

$$\widetilde{P}_s^{(2)}(\eta, t; \eta, \tau) = \begin{cases} \widetilde{P}_s^{(1)}(\eta, \tau) \ln \dfrac{\eta^2 |x(t) - x(\tau)|^2}{\frac{4}{e} \sin^2 \frac{t-\tau}{2}}, & t \neq \tau, \\ \widetilde{P}_s^{(1)}(\eta, t) \ln(e\eta^2 |x'(t)|^2), & t = \tau \end{cases}$$

with $\widetilde{P}_s^{(1)}(\eta, \tau) = \dfrac{1}{2} J(\eta, \tau)$. Having thses functions, corresponding quadraures can be applied to each of integrals.

4. Numerical integration

For the numerical integration we apply the following interpolation quadrature rules

$$\frac{1}{2\pi} \int_0^{2\pi} f(\tau)\, d\tau \approx \frac{1}{2n} \sum_{j=0}^{2n-1} f(t_j), \tag{16}$$

$$\frac{1}{2\pi} \int_0^{2\pi} f(\tau) \ln \left(\frac{4}{e} \sin^2 t - \frac{\tau}{2} \right) d\tau \approx \sum_{j=0}^{2n-1} f(t_j) F_j(t), \tag{17}$$

$$\frac{1}{2\pi} \int\limits_{\Pi} g(\xi, \tau) d\tau d\xi \approx \frac{1}{2n} \sum_{k=1}^{N} \sum_{j=0}^{2n-1} \alpha_k g(\eta_k, t_j), \tag{18}$$

$$\frac{1}{2\pi} \int\limits_{\Pi} g(\xi, \tau) \cot \frac{\tau - t}{2} d\tau d\xi \approx \sum_{k=1}^{N} \sum_{j=0}^{2n-1} \alpha_k g(\eta_k, t_j) T_j(t), \tag{19}$$

$$\frac{1}{2\pi} \int\limits_{\Pi} g(\xi, \tau) \ln \left(\frac{4}{e} \sin^2 t - \frac{\tau}{2} \right) d\tau d\xi \approx \sum_{k=1}^{N} \sum_{j=0}^{2n-1} \alpha_k g(\eta_k, t_j) F_j(t) \tag{20}$$

with quadrature weights $\alpha_k \in \mathbb{R}$, quadrature points $\eta_k \in (0, 1)$, $k = 1, \ldots, N$ and $t_j = \frac{j\pi}{n}$, $j = 0, \ldots, 2n - 1$, and the weight functions

$$\begin{aligned} F_j(t) &= -\frac{1}{2n} \left(1 + 2 \sum_{m=1}^{n-1} \frac{1}{m} \cos m(t - t_j) + \frac{1}{n} \cos n(t - t_j) \right), \\ T_j(t) &= -\frac{1}{n} \sum_{m=1}^{n-1} \sin m(t - t_j) - \frac{1}{2n} \sin n(t - t_j). \end{aligned}$$

Note, that η_k are quadrature nodes of some open quadrature. We use a midpoint rectangles quadrature with respect to the variable ξ, so $\alpha_k = 1/N$ and $\eta_k = 0.5(2k - 1)/N$ for $k = 1, \ldots N$. For 2π-periodic integrals we employ the trapezoidal quadrature rule based on trigonometric interpolation with equidistant points t_j. For boundary integrals and their corresponding parameterizations we use formulas (16)-(17) while for double integrals we apply (18)-(20). Thus for three methods, we have the following approximations:

– The approach based on Green's theorem

$$I_1(x(t)) \approx \sum_{j=0}^{2n-1} \Big[H_1^{(1)}(t;t_j)F_j(t) + \frac{1}{2n} H_1^{(2)}(t,t_j) \Big], \quad t \in [0,2\pi], \tag{21}$$

$$I_2(x) \approx \frac{1}{2n} \sum_{j=0}^{2n-1} H_2(x;t_j), \quad x \in D, \tag{22}$$

$$I_3(x) \approx \frac{1}{2n} \sum_{j=0}^{2n-1} \Big[H_3^{(1)}(x;t_j) + H_3^{(2)}(x;t_j) + H_3^{(3)}(x;t_j) \Big], \quad x \in D. \tag{23}$$

– The RIM

$$I_1(x(t)) \approx \frac{1}{2n} \sum_{j=0}^{2n-1} S_1(t;t_j), \quad t \in [0,2\pi], \tag{24}$$

$$I_2(x) \approx \frac{1}{2n} \sum_{j=0}^{2n-1} S_2(x;t_j), \quad x \in D, \tag{25}$$

$$I_3(x) \approx \frac{1}{2n} \sum_{j=0}^{2n-1} S_3(x;t_j), \quad x \in D. \tag{26}$$

– The quadratures-based approach

$$I_1(x(t)) \approx \frac{1}{N} \sum_{k=1}^{N} \sum_{j=0}^{2n-1} \widehat{P}_s(t;\eta_k,t_j), \quad t \in [0,2\pi], \tag{27}$$

$$I_2(p(\eta,t)) \approx \frac{1}{N} \sum_{k=1}^{N} \sum_{j=0}^{2n-1} \overline{R}_s(\eta,t;\eta_k,t_j), \quad (\eta,t) \in \Pi, \tag{28}$$

$$I_3(p(\eta,t)) \approx \frac{1}{N} \sum_{k=1}^{N} \sum_{j=0}^{2n-1} \overline{P}_s(\eta,t;\eta_k,t_j), \quad (\eta,t) \in \Pi, \tag{29}$$

where

$$\overline{R}_s(\eta,t;\eta_k,t_j) = \begin{cases} \dfrac{1}{2n} \widetilde{R}_s(\eta,t;\eta_k,t_j) & \text{for} \quad \eta \neq \eta_k, \\[2ex] \dfrac{1}{2n} \widetilde{R}_s^{(1)}(\eta,t;\eta_k,t_j) + T_j(t) \widetilde{R}_s^{(2)}(\eta,t;\eta_k,t_j) & \text{for} \quad \eta = \eta_k, \end{cases}$$

$$\overline{P}_s(\eta,t;\eta_k,t_j) = \begin{cases} \dfrac{1}{2n} \widetilde{P}_s(\eta,t;\eta_k,t_j) & \text{for} \quad \eta \neq \eta_k, \\[2ex] \widetilde{P}_s^{(1)}(\eta,t;\eta_k,t_j) F_j(t) + \dfrac{1}{2n} \widetilde{P}_s^{(2)}(\eta,t;\eta_k,t_j) & \text{for} \quad \eta = \eta_k. \end{cases}$$

5. Numerical experiments

The numerical results for integrals I_1, I_2, I_3 considering two different domains for each of them are presented. There will be provided numerical integration results for the approach based on Green's theorem (see formulas (21)-(23)) and straightforward calculation of double integrals based on the quadratures application (formulas (27)-(29)). Using the RIM domain integrals are reduced to boundary integrals and after parameterization are calculated using the formulas (24)-(26).

Calculation of I_1.

Example 1.1. Let the domain D is bounded by the circle with radius equals 1:

$$\Gamma = \{x(t) = (\cos(t), \sin(t)),\ t \in [0, 2\pi]\}.$$

Let's calculate $I_1(x)$ at point $x = x^* = (1, 0)$ in the Cartesian coordinate system which in representation by the formula (5) corresponds to the parameter $t = 0$.

$$I_1(x^*) = \frac{1}{2\pi} \iint\limits_D \frac{(1 - y_1)}{(1 - y_1)^2 + y_2^2} dy_1 dy_2 = \frac{1}{2}.$$

Applying the RIM we obtain the following boundary integral

$$I_1(x^*) = \frac{1}{2\pi} \int\limits_\Gamma \frac{1}{r} \frac{\partial r}{\partial \nu} (-r_1) ds(y).$$

In Table 1 absolute errors for different discretization parameters n and N are provided for three approaches: based on Green's theorem, the RIM and the quadratures-based method for double integral calculation. Note, that only the quadratures-based approach depends on both parameters.

For the first approach, the exponential rate of convergence is expected. The RIM yields exact value of the integral for different values of n, so the error equals zero. For the last approach, convergence to the exact integral value is not observed with respect to N in the case of the fixed n. If the parameter N is fixed then n increasing decreases the error.

Example 1.2. The domain D is bounded by the curve Γ (see Fig. 1)

$$\Gamma = \{x(t) = (\cos(t), 1 + \sin(t) - \sin^2(t)),\ t \in [0, 2\pi]\},$$

Let's calculate $I_1(x)$ at point $x = x^* = (1, 1)$ that corresponds to the parameter $t = 0$. As the exact value we take 0.4013072210380685.

For all three methods, the behaviour of the absolute errors in Table 2 is similar to the Ex. 1.1. Therefore, according to these two examples, the most efficient method is the first one or the second one with the exponential rate of convergence.

Calculation of I_2.

Example 2.1 The boundary is the same as in Ex. 1.1, $x^* = (\frac{1}{2}, 0)$ and $\sigma(x) = 2 + x_1^2 + x_2^2$. Note that the exact integral value is 0.25 and $x^* = p(\frac{1}{2}, 0)$. The absolute errors are displayed in Table 3.

TABL. 1. Absolute errors for three methods for Ex. 1.1

n	N	$Green's$	RIM	$Quadratures$
16	3	1.11E-16	0.00	8.15E-04
	4	-	-	3.09E-03
	7	-	-	1.37E-02
	8	-	-	1.71E-02
	15	-	-	3.52E-02
32	3	0.00	0.00	2.38E-06
	4	-	-	4.25E-05
	7	-	-	1.17E-03
	8	-	-	1.91E-03
	15	-	-	8.38E-03
64	3	2.22E-16	0.00	2.03E-11
	4	-	-	8.26E-09
	7	-	-	1.01E-05
	8	-	-	3.03E-05
	15	-	-	8.52E-04

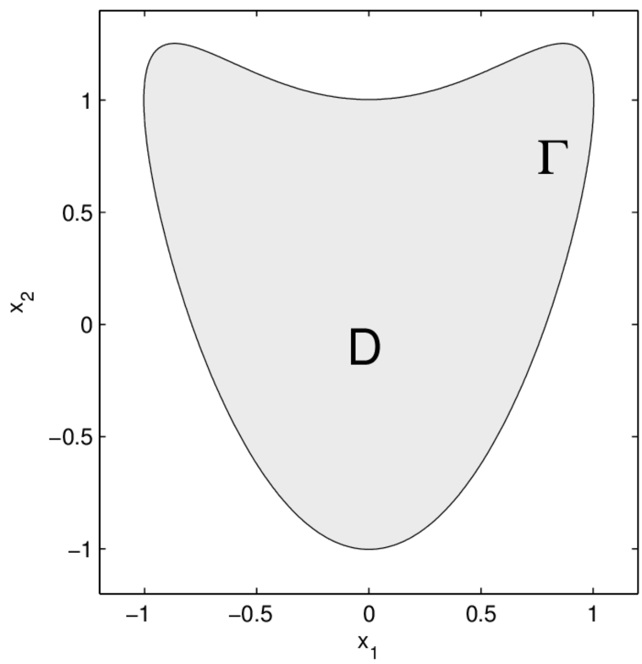

FIG. 1. The domain D in Ex. 1.2

Example 2.2 Let the domain D (see Fig. 2) is defined by the Γ that provided below

$$\Gamma = \{x(t) = (2\cos(t), \sin(t) + \sin^2(t) - 1),\ t \in [0, 2\pi]\},$$

TABL. 2. Absolute errors for three methods for Ex. 1.2

n	N	$Green's$	RIM	$Quadratures$
16	3	3.45E-09	9.57E-08	7.84E-03
	4	-	-	9.55E-04
	7	-	-	4.95E-03
	8	-	-	5.26E-03
	15	-	-	2.28E-03
32	3	1.89E-15	3.35E-14	7.80E-03
	4	-	-	1.69E-04
	7	-	-	7.41E-04
	8	-	-	1.27E-04
	15	-	-	3.70E-03
64	3	2.78E-16	3.89E-16	7.80E-03
	4	-	-	1.72E-04
	7	-	-	8.48E-04
	8	-	-	5.41E-04
	15	-	-	1.61E-04

TABL. 3. Absolute errors for three methods for Ex. 2.1

n	N	$Green's$	RIM	$Quadratures$
16	3	9.27E-12	1.75E-10	2.78E-02
	4	-	-	2.29E-04
	7	-	-	5.04E-03
	8	-	-	1.78E-03
	15	-	-	2.65E-04
32	3	5.55E-17	5.55E-17	2.78E-02
	4	-	-	1.94E-07
	7	-	-	5.10E-03
	8	-	-	5.37E-05
	15	-	-	1.09E-03
64	3	1.11E-16	1.66E-16	2.78E-02
	4	-	-	1.23E-13
	7	-	-	5.10E-03
	8	-	-	3.57E-08
	15	-	-	1.11E-03

Let $\sigma(x) = e^{4x_1x_2}$ and $x^* = (0.5\sqrt{3}), -0.125)$ which means that $(\eta, t) = (0.5, \pi/6)$. As the exact value we take 0.063480913003424. The RIM boundary integral is

$$I_2(x^*) = \frac{1}{2\pi} \int\limits_{\Gamma} \frac{1}{r} \frac{\partial r}{\partial \nu} (0.5e^{-0.25\sqrt{3}} r_1 - 2\sqrt{3}e^{-0.25\sqrt{3}} r_2) ds(y).$$

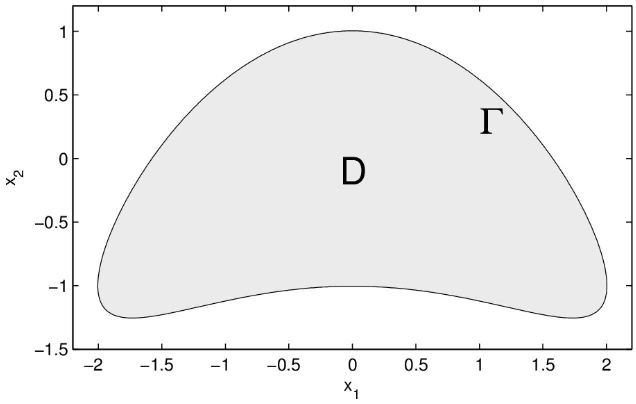

FIG. 2. The domain D in Ex. 2.2

TABL. 4. Absolute errors for three methods for Ex. 2.2

n	N	*Green's*	*RIM*	*Quadratures*
16	3	2.22E-06	7.35E-05	5.72E-02
	4	-	-	7.17E-03
	7	-	-	9.79E-03
	8	-	-	1.26E-02
	15	-	-	1.80E-03
32	3	1.08E-10	1.55E-09	5.76E-02
	4	-	-	1.84E-02
	7	-	-	5.51E-03
	8	-	-	6.25E-03
	15	-	-	6.85E-04
64	3	7.65E-15	7.34E-15	5.76E-02
	4	-	-	1.85E-02
	7	-	-	5.53E-03
	8	-	-	3.84E-03
	15	-	-	1.28E-03

For the RIM and the approach based on Green's theorem, the exponential rate of convergence is observed. Regarding the last approach: with respect to odd N values the convergence is present. It may be explained by the fact that for odd N the quadrature point in (28) coincides with x^* and the singularity is handled.

Calculation of I_3.

Example 3.1. The domain is the same as in Ex. 1.1. Let's calculate $I_3(x)$ at point $x = x^* = (0.5, 0.5)$ in the Cartesian coordinate system and that corresponds to the parameters $\eta = \frac{1}{\sqrt{2}}$, $t = \frac{\pi}{4}$. The exact integral value is -0.125. The numerical results presented in Table 5.

Tabl. 5. Absolute errors for three methods for Ex. 3.1

n	N	$Green's$	RIM	$Quadratures$
16	3	8.66E-04	5.76E-08	2.71E-03
	4	-	-	1.79E-03
	7	-	-	5.74E-04
	8	-	-	2.57E-03
	15	-	-	2.46E-03
32	3	1.93E-04	4.47E-13	2.66E-03
	4	-	-	1.68E-03
	7	-	-	3.17E-04
	8	-	-	1.37E-03
	15	-	-	9.52E-04
64	3	5.12E-05	1.39E-17	2.66E-03
	4	-	-	1.68E-03
	7	-	-	3.11E-04
	8	-	-	1.14E-03
	15	-	-	5.23E-04

Example 3.2. Let D is an ellipse with the boundary

$$\Gamma = \{x(t) = (2\cos(t), \sin(t)),\ t \in [0, 2\pi]\}.$$

We calculate $I_3(x)$ at point $x = x^* = (0, 0.5)$ $(x^* = p(0.5, \frac{\pi}{2}))$. As the exact value we take -0.011201558293121.

Tabl. 6. Absolute errors for three methods for Ex. 3.2

n	N	$Green's$	RIM	$Quadratures$
16	3	6.01E-03	3.15E-06	2.95E-02
	4	-	-	8.95E-03
	7	-	-	5.47E-03
	8	-	-	3.34E-03
	15	-	-	1.83E-03
32	3	1.51E-03	5.99E-10	2.95E-03
	4	-	-	8.53E-03
	7	-	-	5.29E-03
	8	-	-	2.17E-03
	15	-	-	1.18E-03
64	3	3.76E-04	2.65E-10	2.95E-03
	4	-	-	8.53E-03
	7	-	-	5.29E-03
	8	-	-	2.09E-03
	15	-	-	1.15E-03

Following the results in the last two examples, the best method for numerical integration of I_3 is the RIM and its exponential convergence after quadrature application. The approach based on Green's theorem is being converged as well as the quadrature-based method, but not so fast as the RIM. Hence, the RIM is the most effective approach to calculate the integral I_3.

6. Conclusions

In this article, the numerical integration of singular double integrals based on Green's theorem together with the RIM and quadratures-based method has been considered. The numerical results for each of these three methods for different examples have been provided. As a conclusion, we can point out that the advantage of the quadratures-based approach is that, in general, it can be easily applied to the integrals with known and unknown integrands. However, this method requires more calculations and the singularities should be handled in a proper way. The main advantage in the RIM and in the approach that uses Green's theorem is that we have a deal with a boundary integral and applying appropriate quadrature rules to the integrals with smooth enough function it is possible to achieve exponential convergence. Also, the RIM is applicable for any countered curves and can be used for three-dimensional integrals. Numerical integration in doubly connected domains or with unknown integrands may be a theme for further investigation.

Bibliography

1. AL-Jawary M.A. Boundary element formulations for the numerical solution of two-dimensional diffusion problems with variable coefficients / M.A. AL-Jawary, J. Ravnik L.C. Wrobel, L. Škerget // Computers and Mathematics with Applications.– 2012.– Vol. 64.– P. 2695-2711.
2. AL-Jawary M.A. Radial integration boundary integral and integro-differential equation methods for two-dimensional heat conduction problems with variable coefficients / M.A. AL-Jawary, L.C. Wrobel // Engeneering Analysis with Boundary Elements.– 2012.– Vol. 36 (5).– P. 685-695.
3. Beshley A. An integral equation method for the numerical solution of a Dirichlet problem for second-order elliptic equations with variable coefficients / A. Beshley, R. Chapko, B.T. Johansson // Journal of Engineering Mathematics.– 2018.– Vol. 112 (1).– P. 63-73.
4. Beshley A. On the integral equation approach for solution of a Neumann boundary value problem for an elliptic equation with variable coefficients / A. Beshley // Visnyk of the Lviv University. Series Applied Mathematics and Informatics.– 2018.– Vol. 26.– P. 9-19. (in Ukrainian).
5. Dufera T.T. Analysis of Boundary-Domain Integral Equations for Variable-Coefficient Dirichlet BVP in 2D / T.T. Dufera, S.E. Mikhailov // In: *Integral Methods in Science and Engineering: Theoretical and Computational Advances. C. Constanda and A. Kirsh, eds., Springer (Birkhäuser): Boston.*– 2015.– P. 163-175.
6. Fichtengolz G. Course of differential and integral calculus / G. Fichtengolz.– Moscow: Science, 1966.– Vol. 3.
7. Gao X.-W. A boundary element method without internal cells for two-dimensional and three-dimensional elastoplastic problems / X.-W. Gao // Journal of Applied Mechanics (ASME).– 2002.– Vol. 69.– P. 154-160.
8. Gao X.-W. Numerical evaluation of arbitrary singular domain integrals based on radial integration method / X.-W. Gao, H.-F. Peng, // Engineering Analysis with Boundary Elements.– 2011.– Vol. 35 (3).– P. 587-593.
9. Gao X.-W. The radial integration method for evaluation of domain integrals with boundary-only discretization / X.-W. Gao // Engineering Analysis with Boundary Elements.– 2002.– Vol. 26.– P. 905-916.

10. Hu J.-X. Numerical Evaluation of Arbitrary Singular Domain Integrals Using Third-Degree B-Spline Basis Functions / J.-X. Hu, H.-F. Peng, X.-W. Gao // Mathematical Problems in Engineering. – 2014. – Vol. 2014. – P. 10.
11. Kress R. Linear Integral Equations (3rd ed.) / R. Kress. – New-York: Springer-Verlag, 2014.
12. Kress R. On Trefftz' integral equation for the Bernoulli free boundary value problem / R. Kress // Numerische Mathematik. – 2017. – Vol. 136. – P. 503-522.
13. Mikhailov S.E. Localized boundary-domain integral formulations for problems with variable coefficients / S.E. Mikhailov // Engineering Analysis with Boundary Elements. – 2002. – Vol. 26. – P. 681-690.

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