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ON THE NUMERICAL SOLUTION OF A MIXED BOUNDARY VALUE PROBLEM FOR THE ELLIPTIC EQUATION WITH VARIABLE COEFFICIENTS IN DOUBLY CONNECTED PLANAR DOMAINS

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Резюме. Ми розглядаємо чисельне розв'язування мішаної задачі для еліптичного рівняння другого порядку зі змінними коефіцієнтами у двозв'язній області. Розв'язок задачі подається у вигляді суми потенціалів з невідомими густинами і функцією Леві у якості ядра. Підставляючи подання розв'язку в основне рівняння та дві крайові умови, ми отримуємо систему гранично-просторових інтегральних рівнянь. Заміна змінних приводить до параметризованої системи, яка трансформується у систему лінійних алгебричних рівнянь після застосування квадратур та колокації апроксимаційних рівнянь у відповідних вузлах. Наприкінці наведено деякі чисельні результати.

Abstract. We consider a numerical solution of a mixed boundary value problem for the second-order elliptic equation with variable coefficients in a doubly connected domain. A solution of the problem is represented as a sum of potentials with unknown densities and Levi function as a kernel. Substituting the solution representation in the main equation and two boundary conditions we obtain a system of boundary-domain integral equations. The change of variables leads to the parameterised system which is being transformed in a system of linear algebraic equations after quadratures application and collocation of the approximating equations at appropriate points. Some numerical results are provided at the end.

1. Introduction

The elliptic differential equations with variable coefficients are widely spread in many problems of mathematical physics. The coefficients presented in a differential operator mostly correspond to the specific material parameters (for instance, thermal, electrical or hydraulic conductivity) of a considered physical process.

There are well-known effective methods (the boundary element method, the boundary integral equation method) for solving problems defined in bounded or infinite domains. The main advantage of these approaches is decreasing of the dimension of the problem – the solution in a domain can be represented using specific expression only over the boundary. However, in this case, a fundamental solution for a general differential operator is required. Unfortunately, a fundamental solution, in general, is unknown for differential equations with

Key words. Elliptic equation with variable coefficients, mixed boundary value problem, parametrix, boundary-domain integral equations, quadrature formulas.

variable coefficients or its finding can be quite complicated (in contrast to equations with constant coefficients). Therefore, efficient methods to solve such kind of problems are welcomed.

One of the approaches that has been proposed for the numerical solution of so-called the generalized Laplace equation [9] (a second-order linear elliptic partial differential equation with variable coefficients) is described in [10]. The main idea is to transform the starting equation with variable coefficients into a constant-coefficient equation for which a fundamental solution is available and then any of mentioned above effective methods can be applied. The first step in the procedure is to avoid the first partial derivatives of the unknown function and next step is to approximate the transformed equation using constant coefficients.

It is not mandatory to obtain the constant-coefficient equation to solve the problem. As an example, in [1] for solving a two-dimensional mixed problem (where the Dirichlet condition prescribed on a part of the boundary and the Neumann condition prescribed on the remaining part of the domain boundary) with variable coefficients a special function (parametrix) has been used in the Green formula to reduce the initial boundary value problem to a boundary-domain integral equation or boundary-domain integro-differential-equation with the following discretisation of the domain and application of the collocation method. Another similar technique for solving this problem, but with using the radial integration method [5], has been proposed in [2]. The radial integration method was employed to convert domain integrals into equivalent boundary integrals.

In this paper, we consider the numerical solution of a mixed boundary value problem in a doubly connected domain where the Neumann condition is defined on the outer boundary, meanwhile as the Dirichlet condition prescribed on the inner boundary.

Let D_0 be a simple bounded domain in $\mathbb{R}^2$ with boundary $\Gamma_0 \in C^2$. Let D_{-1} be a domain bounded by curve $\Gamma_{-1} \in C^2$ and $\overline{D}_{-1} \subset D_0$. We define that $D = D_0 \setminus \overline{D}_{-1}$. We consider the following mixed boundary value problem in the doubly connected planar domain D for elliptic equation with variable coefficients: need to find function $u \in H^1(D)$ that satisfies the differential equation

$$Lu(x) = \operatorname{div}(\sigma(x) \operatorname{grad} u(x)) = 0, \quad x \in D, \tag{1}$$

the Dirichlet condition on Γ_{-1}

$$u = f_1 \ \text{ on } \ \Gamma_{-1} \tag{2}$$

and the Neumann condition on Γ_0

$$\sigma \frac{\partial u}{\partial \nu} = f_2 \ \text{ on } \ \Gamma_0. \tag{3}$$

Here, $\sigma \in C^\infty(\overline{D})$, $\sigma > 0$, f_1, f_2 are known functions and ν is the outward unit normal to the boundary.

This problem can be interpreted as a stationary heat transfer problem in an isotropic medium for a two-dimensional bounded body with prescribed temperature and heat flux on different boundaries. Since the main equation is homogeneous we assume that a heat source is not available. The function $\sigma(x)$, in this case, is a known thermal conductivity.

For the outline of the work, in Section 2, we reduce our differential problem to a system of boundary-domain integral equations, obtain an equivalent system in a parameterised form and split singularities from some kernels. A full discretisation of the system with applied quadratures and approximation formula of the solution in a domain are presented in Section 3. In Section 4, two numerical examples for different domain configurations are considered. Some conclusions are given in Section 5.

2. Reduction to a system of boundary-domain integral equations

As it was mentioned above, there is no ability to reduce the problem to a boundary integral equation as a fundamental solution is not available in the explicit form, in general case, for elliptic equations with variable coefficients. But, we can use a parametrix to work only with integrals instead of the differential equation and boundary conditions, however, it leads to domain integrals appearing. A parametrix (or Levi function) $P(x,y)$, $x, y \in \mathbb{R}^2$ should satisfy the following expression [8]

$$L_x P(x,y) = \delta(x-y) + R(x,y), \tag{4}$$

where δ is the Dirac function and the remainder function R has a weak singularity for $x = y$. In the two-dimensional case we can define the parametrix as the fundamental solution with frozen coefficients $a(x) = a(y)$ corresponding to the operator L, i.e., in the form

$$P(x,y) = \frac{\ln|x-y|}{2\pi\sigma(y)}, \quad x, y \in \mathbb{R}^2, \;\; x \neq y$$

with the remainder function

$$R(x,y) = \frac{(x-y)\cdot \operatorname{grad}\sigma(x)}{2\pi\sigma(y)|x-y|^2}, \quad x, y \in \mathbb{R}^2 \;\; x \neq y.$$

It is not difficult to verify that functions $P(x,y)$ and $R(x,y)$ satisfy (4). Should note that the parametrix function is not unique.

We seek the solution as a sum of potentials, but instead of the fundamental solution of the differential operator we use the Levi function

$$\begin{aligned} u(x) = &\int\limits_D \psi(y) P(x,y)\,dy + \int\limits_{\Gamma_{-1}} \psi_{-1}(y) P(x,y)\,ds(y) + \\ &+ \int\limits_{\Gamma_0} \psi_0(y) P(x,y)\,ds(y), \quad x \in D, \end{aligned} \tag{5}$$

where $\psi \in C(D)$, $\psi_{-1} \in C(\Gamma_{-1})$ and $\psi_0 \in C(\Gamma_0)$ are unknown densities. Substituting (5) in (1)-(3) we obtain the following system of a boundary-domain integral equations

$$
\begin{cases}
\psi(x) + \int\limits_D \psi(y) R(x,y)\,dy + \int\limits_{\Gamma_{-1}} \psi_{-1}(y) R(x,y)\,ds(y) + \\
\qquad + \int\limits_{\Gamma_0} \psi_0(y) R(x,y)\,ds(y) = 0, \;\; x \in D, \\[2ex]
\int\limits_D \psi(y) P(x,y)\,dy + \int\limits_{\Gamma_{-1}} \psi_{-1}(y) P(x,y)\,ds(y) + \\
\qquad + \int\limits_{\Gamma_0} \psi_0(y) P(x,y)\,ds(y) = f_1(x), \;\; x \in \Gamma_{-1}, \\[2ex]
-\frac{1}{2}\psi_0(x) + \int\limits_D \psi(y)\sigma(x) \frac{\partial P(x,y)}{\partial \nu(x)}\,dy + \\
\qquad + \int\limits_{\Gamma_{-1}} \psi_{-1}(y)\sigma(x) \frac{\partial P(x,y)}{\partial \nu(x)}\,ds(y) + \\
\qquad + \int\limits_{\Gamma_0} \psi_0(y)\sigma(x) \frac{\partial P(x,y)}{\partial \nu(x)}\,ds(y) = f_2(x), \quad x \in \Gamma_0.
\end{cases} \tag{6}
$$

If $\sigma(x) = 1$ then the density $\psi(x)$ vanishes (together with domain integrals) and the system is being simplified to a system of boundary integral equations that correspond to the Laplace equation. The similar system for this case can be found in [4].

Let D is symmetric relative to the origin and assume that the closed boundary curves Γ_0, Γ_{-1} are homothetic with factor ξ_{-1} and have the following representations

$$
\begin{aligned}
\Gamma_0 &= \{x(t) = (x_1(t), x_2(t)), \; t \in [0, 2\pi)\}, \\
\Gamma_{-1} &= \{x_{-1}(t) = (\xi_{-1} x_1(t), \xi_{-1} x_2(t)), \; t \in [0, 2\pi)\},
\end{aligned} \tag{7}
$$

where ξ_{-1} is a fixed parameter and $0 < \xi_{-1} < 1$. To obtain the system in the parametrized form we use the change of variables in the integrals over domain in (6)

$$
y_1 = p_1(\xi, \tau) = \xi x_1(\tau),
$$

$$
y_2 = p_2(\xi, \tau) = \xi x_2(\tau),
$$

where $(\xi, \tau) \in \Pi = (\xi_{-1}, 1) \times [0, 2\pi)$ and Jacobian $J(\xi, \tau) = \xi(x_1(\tau) x_2'(\tau) - x_2(\tau) x_1'(\tau))$. The notation $p = (p_1, p_2)$ is used for the function mapping into Π.

This yields the following system

$$\begin{cases} \varphi(\eta,t)+\dfrac{1}{2\pi}\displaystyle\int\limits_{\Pi}\varphi(\xi,\tau)\widetilde{R}(\eta,t;\xi,\tau)\,d\tau d\xi+ \\ \qquad +\dfrac{1}{2\pi}\displaystyle\int\limits_{0}^{2\pi}\varphi_{-1}(\xi_{-1},\tau)\widetilde{R}_{-1}(\eta,t;\xi_{-1},\tau)\,d\tau+ \\ \qquad +\dfrac{1}{2\pi}\displaystyle\int\limits_{0}^{2\pi}\varphi_0(\tau)\widetilde{R}_0(\eta,t;\tau)\,d\tau=0,\ (\eta,t)\in\Pi, \\ \dfrac{1}{2\pi}\displaystyle\int\limits_{\Pi}\varphi(\xi,\tau)\check{P}(\xi_{-1},t;\xi,\tau)\,d\tau d\xi+ \\ \qquad +\dfrac{1}{2\pi}\displaystyle\int\limits_{0}^{2\pi}\varphi_{-1}(\xi_{-1},\tau)\check{P}_{-1}(\xi_{-1},t;\xi_{-1},\tau)\,d\tau+ \\ \qquad +\dfrac{1}{2\pi}\displaystyle\int\limits_{0}^{2\pi}\varphi_0(\tau)\check{P}_0(\xi_{-1},t;\tau)\,d\tau=\widetilde{f}_1(\xi_{-1},t),\ t\in[0,2\pi), \\ -\dfrac{1}{2}\varphi_0(t)+\dfrac{1}{2\pi}\displaystyle\int\limits_{\Pi}\varphi(\xi,\tau)\widehat{P}(t;\xi,\tau)\,d\tau d\xi+ \\ \qquad +\dfrac{1}{2\pi}\displaystyle\int\limits_{0}^{2\pi}\varphi_{-1}(\xi_{-1},\tau)\widehat{P}_{-1}(t;\xi_{-1},\tau)\,d\tau+ \\ \qquad +\dfrac{1}{2\pi}\displaystyle\int\limits_{0}^{2\pi}\varphi_0(\tau)\widehat{P}_0(t;\tau)\,d\tau=\widetilde{f}_2(t),\ t\in[0,2\pi), \end{cases} \tag{8}$$

with the functions $\varphi(\eta,t)=\psi(p(\eta,t))$, $\varphi_{-1}(t)=\psi_{-1}(x(t))$, $\varphi_0(t)=\psi_0(x(t))$, $\widetilde{f}_1(t)=f_1(x_{-1}(t))$, $\widetilde{f}_2(t)=f_2(x(t))$ and kernels

$$\widetilde{R}(\eta,t;\xi,\tau)=2\pi R(p(\eta,t),p(\xi,\tau))J(\xi,\tau),$$

$$\widetilde{R}_0(\eta,t;\tau)=2\pi R(p(\eta,t),x(\tau))|x'(\tau)|;$$

$$\check{P}(\xi_{-1},t;\xi,\tau)=2\pi P(\xi_{-1}x(t),p(\xi,\tau))J(\xi,\tau),$$

$$\check{P}_0(\xi_{-1},t;\tau)=2\pi P(\xi_{-1}x(t),x(\tau))|x'(\tau)|;$$

$$\widehat{P}(t;\xi,\tau)=2\pi\sigma(x(t))\frac{\partial P(x(t),\xi x(\tau))}{\partial\nu(x(t))}J(\xi,\tau),$$

$$\widehat{P}_0(t;\tau)=2\pi\sigma(x(t))\frac{\partial P(x(t),x(\tau))}{\partial\nu(x(t))}|x'(\tau)|;$$

$$\widetilde{R}_{-1}(\eta,t;\xi_{-1},\tau)=2\pi R(p(\eta,t),\xi_{-1}x(\tau))\xi_{-1}|x'(\tau)|;$$

$$\check{P}_{-1}(\xi_{-1},t;\xi_{-1},\tau)=2\pi P(\xi_{-1}x(t),\xi_{-1}x(\tau))\xi_{-1}|x'(\tau)|;$$

$$\widehat{P}_{-1}(t;\xi_{-1},\tau)=2\pi\sigma(x(t))\frac{\partial P(x(t),\xi_{-1}x(\tau))}{\partial\nu(x(t))}\xi_{-1}|x'(\tau)|.$$

Exploring the kernels it is easy to see that the kernels $\widetilde{R}$ and $\check{P}_{-1}$ have different singularities. The strong singularity in $\widetilde{R}$ can be handled by applying the ideas from [7] (for more details see [3]). The logarithmic singularity in the kernel $\check{P}_{-1}$ can be split [6] as follows

$$\check{P}_{-1}(\xi_{-1},t;\xi_{-1}\tau)=\check{P}^{(1)}_{-1}(\xi_{-1},\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}+\check{P}^{(2)}_{-1}(\xi_{-1},t;\xi_{-1}\tau)\qquad(9)$$

with

$$\check{P}^{(1)}_{-1}(t,\tau)=\frac{1}{2}\frac{\xi_{-1}|x'(\tau)|}{\sigma(\xi_{-1}x(\tau))},$$

and

$$\check{P}^{(2)}_{-1}(t,\tau)=\frac{\xi_{-1}|x'(\tau)|}{\sigma(\xi_{-1}x(\tau))}\begin{cases}\dfrac{1}{2}\ln\dfrac{|\xi_{-1}x(t)-\xi_{-1}x(\tau)|^2}{\frac{4}{e}\sin^2\frac{t-\tau}{2}} & \text{for}\quad t\neq\tau,\\[2ex] \dfrac{1}{2}\ln\left(e|\xi_{-1}x'(t)|^2\right) & \text{for}\quad t=\tau.\end{cases}$$

3. Full discretisation and numerical solution of the system

For solving the system (8) we use the interpolation quadrature rules for continuous integrands and integrands with weight function that corresponds to the specific singularity. For continuous integrands we use

$$\frac{1}{2\pi}\int\limits_{\Pi}g(\xi,\tau)d\tau d\xi\approx\frac{1}{2n}\sum_{k=1}^{N}\sum_{i=0}^{2n-1}\alpha_k g(\eta_k,t_i),\qquad(10)$$

$$\frac{1}{2\pi}\int_0^{2\pi}f(\tau)\,d\tau\approx\frac{1}{2n}\sum_{k=0}^{2n-1}f(t_k).\qquad(11)$$

The following quadratures are used for integrals with strong and logarithmic singularities

$$\frac{1}{2\pi}\int\limits_{\Pi}g(\xi,\tau)\cot\frac{\tau-t}{2}d\tau d\xi\approx\sum_{k=1}^{N}\sum_{i=0}^{2n-1}\alpha_k g(\eta_k,t_i)T_i(t),\qquad(12)$$

$$\frac{1}{2\pi}\int_0^{2\pi}f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)d\tau\approx\sum_{k=0}^{2n-1}f(t_k)\,F_k(t),\qquad(13)$$

In formulas (10), (13) $\alpha_k\in\mathbb{R}^2$ are quadrature weights, $\eta_k\in(0,1)$, $k=1,\dots,N$ – some quadrature points. For 2π-periodic integrals we employ the trapezoidal quadrature rule based on trigonometric interpolation with equidistant points $t_i=i\pi/n,\ i=0,\dots 2n-1,\ n\in\mathbb{N}$. The weight functions $T_i(t)$ and

$F_k(t)$ are defined as follows

$$\begin{aligned} T_i(t) &= -\frac{1}{n}\sum_{m=1}^{n-1} \sin m(t-t_i) - \frac{1}{2n}\sin n(t-t_i), \\ F_k(t) &= -\frac{1}{2n}\left(1 + 2\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_k) + \frac{1}{n}\cos n(t-t_k)\right). \end{aligned}$$

The use of these quadratures in (8) and collocation of the approximating equations at quadrature points lead to the linear system

$$\left\{ \begin{aligned} &\varphi_{mi} + \sum_{k=1}^{N}\sum_{j=0}^{2n-1}\alpha_k\varphi_{kj}\bar{R}(\eta_m, t_i; \eta_k, t_j) + \\ &\qquad + \frac{1}{2n}\sum_{j=0}^{2n-1}\varphi_{-1j}\widetilde{R}_{-1}(\eta_m, t_i; \xi_{-1}, t_j) + \\ &\qquad + \sum_{j=0}^{2n-1}\varphi_{0j}\widetilde{R}_0(\eta_m, t_i; t_j) = 0, \\ &\frac{1}{2n}\sum_{k=1}^{N}\sum_{j=0}^{2n-1}\alpha_k\varphi_{kj}\check{P}(\xi_{-1}, t_i; \eta_k, t_j) + \frac{1}{2n}\sum_{j=0}^{2n-1}\varphi_{0j}\check{P}_0(\xi_{-1}, t_i, t_j) + \\ &+ \sum_{j=0}^{2n-1}\varphi_{-1j}\left[\check{P}_{-1}^{(1)}(\xi_{-1}, t_j)F_j(t_i) + \frac{1}{2n}\check{P}_{-1}^{(2)}(\xi_{-1}, t_i; \xi_{-1}, t_j)\right] = \widetilde{f}_{1i}, \\ &-\frac{1}{2}\widetilde{\varphi}_{0i} + \frac{1}{2n}\sum_{k=1}^{N}\sum_{j=0}^{2n-1}\alpha_k\varphi_{kj}\widehat{P}(t_i; \eta_k, t_j) + \\ &\qquad + \frac{1}{2n}\sum_{j=0}^{2n-1}\varphi_{-1j}\widehat{P}_{-1}(t_i; \xi_{-1}, t_j) + \\ &\qquad + \frac{1}{2n}\sum_{j=0}^{2n-1}\varphi_{0j}\widehat{P}_0(t_i, t_j) = \widetilde{f}_{2i}, \end{aligned} \right. \tag{14}$$

with

$$\bar{R}(\eta_m, t_i; \eta_k, t_j) = \begin{cases} \frac{1}{2n}\widetilde{R}(\eta_m, t_i; \eta_k, t_j) & \text{for} \quad m \neq k, \\ \frac{1}{2n}\widetilde{R}^{(1)}(\eta_m, t_i; \eta_k, t_j) + T_j(t)\widetilde{R}^{(2)}(\eta_m, t_i; \eta_k, t_j) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad m = k, \end{cases}$$

and the right-hand side $\widetilde{f}_{1i} = \widetilde{f}_1(t_i)$ and $\widetilde{f}_{2i} = \widetilde{f}_2(t_i)$.

Here, we use the following notation $\varphi_{mi} \approx \varphi(\eta_m, t_i)$, $\varphi_{-1i} \approx \varphi_{-1}(t_i)$ and $\varphi_{0i} \approx \varphi_0(t_i)$ for $m = 1, \ldots, N$ and $i = 0, \ldots, 2n-1$. The kernels $\widetilde{R}^{(1)}$ and $\widetilde{R}^{(2)}$ are smooth functions and their representations are provided in [3].

Solving the system (14) we obtain the approximate values of unknown densities. Having these values we can find the approximation of the solution (1)-(3) in the domain D using the following formula

$$
\begin{aligned}
u(\eta_m, t_i) \approx \sum_{k=1}^{N} \sum_{j=0}^{2n-1} \alpha_k \varphi_{kj} \overline{P}(\eta_m, t_i; \eta_k, t_j) + \\
+ \frac{1}{2n} \sum_{j=0}^{2n-1} \varphi_{-1j} \widetilde{P}_{-1}(\eta_m, t_i; \xi_{-1}, t_j) + \\
+ \frac{1}{2n} \sum_{j=0}^{2n-1} \varphi_{0j} \widetilde{P}_0(\eta_m, t_i; t_j),
\end{aligned} \tag{15}
$$

with

$$
\overline{P}(\eta_m, t_i; \eta_k, t_j) = \begin{cases} \dfrac{1}{2n} \widetilde{P}(\eta_m, t_i; \eta_k, t_j) & \text{for} \quad m \neq k, \\ \widetilde{P}^{(1)}(\eta_m, t_i; \eta_k, t_j) F_j(t_i) + \dfrac{1}{2n} \widetilde{P}^{(2)}(\eta_m, t_i; \eta_k, t_j) & \text{for} \quad m = k, \end{cases}
$$

where $\widetilde{P}^{(1)}(\eta_m, t_i; \eta_k, t_j)$, $\widetilde{P}^{(2)}(\eta_m, t_i; \eta_k, t_j)$ smooth enough functions.

4. Numerical experiments

In this section, we present some numerical results for two different examples. Together with the approximation of solution in the domain, we will provide numerical results for approximations of the normal derivative on Γ_{-1} (taking into account the jump relations of the single-layer potential normal derivative [6]) and the trace of the solution on Γ_0

$$
\begin{aligned}
\frac{\partial u}{\partial \nu}(x) = -\frac{1}{2}\psi_{-1}(x) + \int_D \psi(y) \frac{\partial P(x,y)}{\partial \nu(x)} \, dy + \int_{\Gamma_{-1}} \psi_{-1}(y) \frac{\partial P(x,y)}{\partial \nu(x)} \, ds(y) + \\
+ \int_{\Gamma_0} \psi_0(y) \frac{\partial P(x,y)}{\partial \nu(x)} \, ds(y), \quad x \in \Gamma_{-1},
\end{aligned}
$$

$$
\begin{aligned}
u(x) = \int_D \psi(y) P(x,y) \, dy + \int_{\Gamma_{-1}} \psi_{-1}(y) P(x,y) \, ds(y) + \\
+ \int_{\Gamma_0} \psi_0(y) P(x,y) \, ds(y), \quad x \in \Gamma_0.
\end{aligned}
$$

Example 1. Let the domain D (see Fig. 1) is bounded by the two circles:

$$
\begin{aligned}
\Gamma_0 = \{x(t) = (1.2\cos(t), 1.2\sin(t)),\ t \in [0, 2\pi)\}, \\
\Gamma_{-1} = \{x_{-1}(t) = (0.6\cos(t), 0.6\sin(t)),\ t \in [0, 2\pi)\}.
\end{aligned}
$$

Here we have $\xi_{-1} = 0.5$. The function σ is given and equal

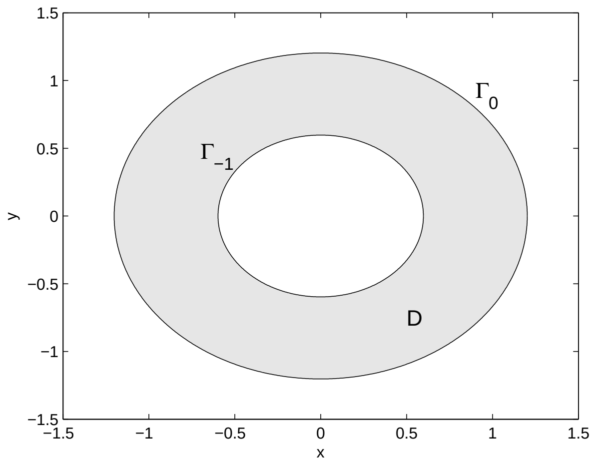


FIG. 1. The solution domain D in Ex. 1

$$\sigma(x) = 4 - x_1^2 + x_2^2, \quad x \in D.$$

Let us choose the boundary functions f_1 and f_2 of the elliptic problem as

$$f_1 = x_1 x_2 \quad \text{on } \Gamma_{-1}, \qquad f_2 = 0.6 x_1 x_2 (4 - x_1^2 + x_2^2) \quad \text{on } \Gamma_0.$$

Easy to verify that $u_{ex} = x_1 x_2$ is the exact solution to (1)-(3).

In (10),(12) we use the midpoint quadrature as a quadrature rule with respect to $\xi \in (\xi_{-1}, 1)$ with weights $\alpha_k = \frac{1-\xi_{-1}}{N}$ and quadrature nodes $\eta_k = 1 - \frac{1-\xi_{-1}}{2N}(2k-1)$, $k = 1, \dots, N$.

TABL. 1. Absolute error on inner curves $\tilde{\Gamma}_1$-$\tilde{\Gamma}_3$ for Ex. 1

N	n	$\Vert u_{Nn} - u_{ex}\Vert_{\infty,\tilde{\Gamma}_1}$	$\Vert u_{Nn} - u_{ex}\Vert_{\infty,\tilde{\Gamma}_2}$	$\Vert u_{Nn} - u_{ex}\Vert_{\infty,\tilde{\Gamma}_3}$
3	32	2.33E-05	6.64E-05	1.31E-04
	64	8.86E-08	2.52E-07	5.47E-07
6	64	1.16E-05	3.45E-05	7.51E-05
	128	4.97E-08	1.47E-07	3.21E-07
12	128	5.80E-06	1.76E-05	3.85E-05
	256	2.63E-08	7.97E-08	1.74E-07

We will provide the numerical error of the proposed approach on three curves within the domain that are homothetic to the outer boundary and have the following parametric representations

$$\tilde{\Gamma}_k : \tilde{x}_k = (\xi_{-1} + \frac{1-\xi_{-1}}{40}(12k-5))x(t), \;\; t \in [0, 2\pi), \;\; k = 1, 2, 3. \qquad (16)$$

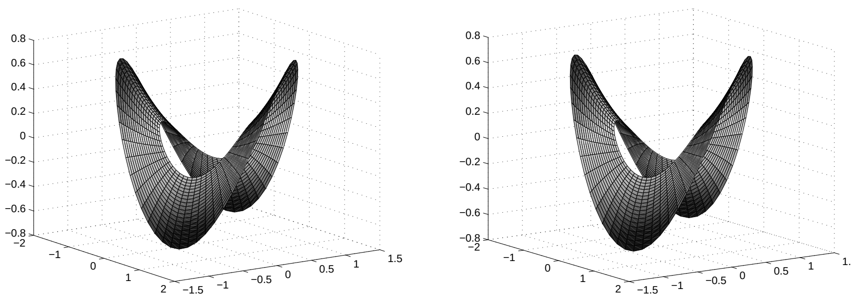
Straightforward calculation gives that homothetic factors related to the curves $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$, $\tilde{\Gamma}_3$ are 0.5875, 0.7375 and 0.8875 respectively. They correspond to the 4th, 10th, 16th curve counting from the first inner curve after Γ_{-1} in case when discretisation parameter $N = 20$. The absolute errors for different discretisation parameters N and n are presented in Table 1.

TABL. 2. Absolute error of the normal derivative and the function on boundaries and relative error in D for Ex. 1

N	n	$\|\frac{\partial u_{Nn}}{\partial \nu} - \frac{\partial u_{ex}}{\partial \nu}\|_{\infty,\Gamma_{-1}}$	$\|u_{Nn} - u_{ex}\|_{\infty,\Gamma_0}$	$\frac{\|u_{Nn}-u_{ex}\|_{L_2(D)}}{\|u_{ex}\|_{L_2(D)}} \cdot 100\%$
3	32	3.09E-04	1.03E-04	1.455
	64	1.17E-06	3.38E-07	0.271
6	64	1.89E-04	5.67E-05	0.270
	128	8.08E-07	2.53E-06	0.025
12	128	1.05E-04	3.37E-04	0.277
	256	4.73E-07	7.98E-07	0.276

In Table 2 we present the absolute errors of the normal derivative on the Γ_{-1} and the solution on the Γ_0 together with relative errors with respect to the L_2-norm in the domain D for the same parameters N and n as in Table 1. To calculate the relative error in the domain we use the following approximation with $\tilde{N} = 20$ and $\tilde{n} = 32$

$$\frac{\|u_{Nn} - u_{ex}\|_{L_2(D)}}{\|u_{ex}\|_{L_2(D)}} \approx \left(\frac{\sum_{k=1}^{\tilde{N}} \sum_{j=0}^{2\tilde{n}-1} (u_{Nn} - u_{ex})^2(\tilde{\eta}_k, \tilde{t}_j) J(\tilde{\eta}_k, \tilde{t}_j)}{\sum_{k=1}^{\tilde{N}} \sum_{j=0}^{2\tilde{n}-1} u_{ex}^2(\tilde{\eta}_k, \tilde{t}_j) J(\tilde{\eta}_k, \tilde{t}_j)} \right)^{1/2}. \quad (17)$$



a). exact solution b). approximate solution

FIG. 2. Exact solution and numerical approximation in domain D for Ex. 1

The numerical approximation (for discretisation parameters $N = 6$, $n = 64$) and the exact solution in the domain D are shown in Fig. 2. From the numerical results, we see that parameters N and n are linked between each other – double increase N requires to increase the parameter n at least by two times to decrease the error. But, in general, presented relative errors in the domain look pretty good as well as absolute errors on inner curves.

Example 2. Let the domain D (see Fig. 3) bounded by the two ellipses:

$$\Gamma_0 = \{x(t) = (a\cos(t), b\sin(t)),\ t \in [0, 2\pi)\},$$
$$\Gamma_{-1} = \{x_{-1}(t) = (0.4a\cos(t), 0.4b\sin(t)),\ t \in [0, 2\pi)\}.$$

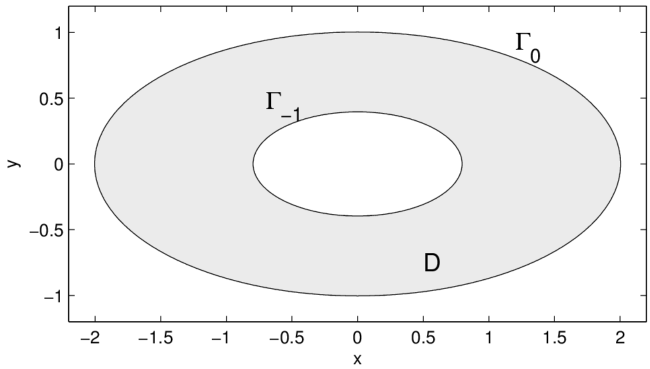


FIG. 3. The solution domain D in Ex. 2

TABL. 3. Absolute error on inner curves $\tilde{\Gamma}_1$-$\tilde{\Gamma}_3$ for Ex. 2

N	n	$\|u_{Nn} - u_{ex}\|_{\infty,\tilde{\Gamma}_1}$	$\|u_{Nn} - u_{ex}\|_{\infty,\tilde{\Gamma}_2}$	$\|u_{Nn} - u_{ex}\|_{\infty,\tilde{\Gamma}_3}$
3	32	3.92E-04	9.38E-04	3.07E-03
	64	3.28E-06	1.05E-05	3.76E-05
6	64	2.16E-04	5.30E-04	1.05E-03
	128	1.99E-06	6.24E-06	1.63E-05
12	128	1.18E-04	2.82E-04	5.46E-04
	256	1.14E-06	3.47E-06	8.72E-06

TABL. 4. Absolute error of the normal derivative and the function on boundaries and relative error in D for Ex. 2

N	n	$\|\frac{\partial \tilde{u}}{\partial \nu} - \frac{\partial u_{ex}}{\partial \nu}\|_{\infty,\Gamma_{-1}}$	$\|\tilde{u} - u_{ex}\|_{\infty,\Gamma_0}$	$\frac{\|u_{Nn}-u_{ex}\|_{L_2(D)}}{\|u_{ex}\|_{L_2(D)}} \cdot 100\%$
3	32	5.60E-03	7.67E-02	1.695
	64	3.10E-05	1.59E-04	0.377
6	64	4.07E-03	2.99E-02	0.377
	128	2.65E-05	1.02E-04	0.052
12	128	2.43E-03	5.54E-02	0.094
	256	1.72E-05	8.45E-04	0.077

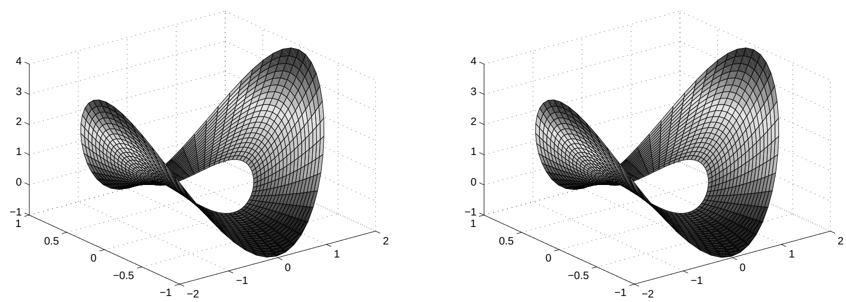
Here we have parameters $a = 2,\ b = 1$ and $\xi_{-1} = 0.4$. The function σ has following representation

$$\sigma(x) = 8 + 2x_1x_2, \quad x \in D.$$

The boundary functions f_1 and f_2 are known

$$f_1 = x_1^2 - x_2^2 \quad \text{on } \Gamma_{-1}, \qquad f_2 = (8 + 2x_1x_2)(x_1^2 - 4x_2^2)(0.25x_1^2 + 4x_2^2)^{-0.5} \quad \text{on } \Gamma_0.$$

For this example, the exact solution is $u_{ex} = x_1^2 - x_2^2$.



a). exact solution   b). approximate solution

FIG. 4. Exact solution and numerical approximation in domain D for Ex. 2

The absolute errors on inner curves (16) are shown in Table 3. Similarly to the Ex. 1., the relative error of the solution in domain D, the absolute errors of its normal derivative on the inner boundary Γ_{-1} and the solution error on the outer boundary Γ_0 are displayed in Table 4. In Fig. 4 the exact solution in the domain D and its approximation for discretisation parameters $N = 6$ and $n = 128$ are shown. Observing the results we can see the same high accuracy of the obtained approximation of the solution as in Ex. 1.

5. Conclusion

An indirect integral equation method (based on the solution representation via potentials with densities and using the Levi function) for the numerical solution of a mixed boundary value problem for the generalized Laplace equation in doubly connected domains was applied. The differential problem is reduced to a system of boundary-domain integral equations. As a doubly connected domain, a domain bounded by two homothetic curves is considered. The change of variables in double integrals, quadrature rules application and the collocation of the obtained approximating equations at quadrature nodes lead to a system of the linear equations. Having calculated approximate values of the unknown densities we can find the approximation of the solution in the domain. Applicability of the proposed approach is confirmed by provided numerical results.

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